ElCoM

ELECTRICAL COMPUTER MECHATRONICS

تقدم لجنة ElCoM الاكاديمية

دفتر لمادة:

تفاضل وتكامل (2)

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lecture 1: Integration by substitution

#First

5/Mar/2022

* $\int f'(x) \cdot g(f(x)) \cdot dx$, $u = f(x) \Rightarrow du = f'(x) \cdot dx \Rightarrow dx = \frac{du}{f'(x)}$

* ~~$\int x^2 \cdot \cos(x^3) \cdot dx$~~ , $u = x^3 \Rightarrow du = 3x^2 \cdot dx \Rightarrow dx = \frac{du}{3x^2}$

$= \frac{1}{3}\sin(x^3) + c$

* $\int 3^{\sin x} \cdot \cos \cdot dx$, $u = \sin x \Rightarrow du = \cos x \cdot dx \Rightarrow dx = \frac{du}{\cos x}$

$= \int 3^u \cdot \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$

$= \frac{3^u}{\ln 3} + c$

$= \frac{3^{\sin x}}{\ln 3} + c$

* $\int x \sin(x^2) \cdot dx$, $u = x^2 \to du = 2x \cdot dx \to dx = \frac{du}{2x}$

$= \int \cancel{x} \sin u \cdot \frac{du}{2\cancel{x}}$

$= \frac{-1}{2}\cos u + c$

$= \frac{-1}{2}\cos(x^2) + c$

* $\int 2x\sqrt{1+x^2} \cdot dx$, $u = 1+x^2 \to du = 2x \cdot dx \to dx = \frac{du}{2x}$

$= \int \cancel{2x}\sqrt{u} \cdot \frac{du}{\cancel{2x}}$

$= \frac{2u^{\frac{3}{2}}}{3} + c$

$= \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c$

* $\int_0^{\frac{\pi}{2}} \sqrt{\cos x} \cdot \sin x \cdot dx$, $u = \cos x \to du = -\sin x \cdot dx \to dx = \frac{-du}{\sin x}$, If $x = 0 \to u = \cos(0) = 1$

If $x = \frac{\pi}{2} \to u = \cos(\frac{\pi}{2}) = 0$

$= \int_1^0 \sqrt{u} \cdot \cancel{\sin x} \cdot \frac{-du}{\cancel{\sin x}}$

$= \frac{-2}{3}\sqrt{u^3}\,\Big]_1^0$

$= \frac{-2}{3}(\sqrt{0} - \sqrt{1})$

$= +\frac{2}{3}$

Five Apple

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lecture1: Integration by substitution 5/Mar/2022

First

* $\int \frac{\sqrt{1+\tan x}}{\cos^2 x} \cdot dx$, $u = 1+\tan x \longrightarrow du = \sec^2 x \, dx \longrightarrow dx = \frac{du}{\sec^2 x}$

$\Rightarrow \int \frac{\sqrt{u}}{\cos^2 x} \cdot \frac{du}{\sec^2 x}$

$= \int \cancel{\sec^2 x} \sqrt{u} \cdot \frac{du}{\cancel{\sec^2 x}}$

$= \frac{2}{3} u^{\frac{3}{2}} + c$

$= \frac{2}{3} (1+\tan x)^{\frac{3}{2}} + c$

* $\int \frac{dx}{x(\ln x)^2}$, $u = \ln x \rightarrow du = \frac{dx}{x} \rightarrow dx = x \cdot du$

$= \int \frac{\cancel{x}}{\cancel{x}(\ln x)^2} \cdot du = \int u^{-2} \cdot du = \frac{u^{-1}}{-1} + c = \frac{-1}{\ln x} + c$ #

* $\int \frac{\cos^2(\ln x)}{x} \cdot dx$, $u = \ln x \longrightarrow du = \frac{1}{x} \cdot dx \longrightarrow dx = x \cdot du$

$\Rightarrow \int \frac{\cos^2 u}{\cancel{x}} \cdot \cancel{x} \cdot du$ (note)

$\int \frac{1}{2} + \frac{\cos 2u}{2} \cdot du$

$= \frac{1}{2} u + \frac{1}{4} \sin(2u) + c$

* $\int_0^1 x\sqrt{1-x} \cdot dx$, $u = 1-x \rightarrow du = -dx \rightarrow dx = -du$, $x = 0 \rightarrow u = 1$
$(x = 1-u)$, $x = 1 \rightarrow u = 0$

$\Rightarrow \int_1^0 x\sqrt{u} \cdot -du$

$= \int_0^1 (1-u)(\sqrt{u}) \cdot du$

$= \int_0^1 (u^{\frac{1}{2}} - u^{\frac{3}{2}}) \cdot du$

$= \frac{2}{3} u^{\frac{3}{2}} - \frac{2}{5} u^{\frac{5}{2}} \Big]_0^1$

$\frac{2}{3}(1)^{\frac{3}{2}} - \frac{2}{5}(1)^{\frac{5}{2}} - (0)$

$\frac{2}{3} - \frac{2}{5} = \frac{4}{15}$ #

note:
$\cos^2 x = \frac{1+\cos 2x}{2}$
$\sin^2 x = \frac{1-\cos 2x}{2}$

lecture 1: Integration by substituation. 5/Mar/2020

#First

$*\int \frac{e^{-x}}{\sqrt{1-e^{-x}}} \cdot dx \quad , u = 1-e^{-x} \to du = +e^{-x} \cdot dx \to dx = \frac{du}{e^{-x}}$

$\Rightarrow \int \frac{\cancel{e^{-x}}}{\sqrt{u}} \cdot \frac{du}{\cancel{e^{-x}}}$

$= 2u^{\frac{1}{2}} + c = 2\sqrt{1-e^{-x}} + c$ #

$*\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}} \cdot dx \quad , u = e^{-x} \to du = -e^{-x} \cdot dx \to dx = \frac{du}{-e^{-x}}$

$(e^{-x})^2$

$\Rightarrow \int \frac{\cancel{u}}{\sqrt{1-u^2}} \cdot \frac{du}{-\cancel{u}}$

$= (-)\sin^{-1} u + c$ or $\cos^{-1} u$

$= -\sin^{-1}(e^{-x}) + c$ #

$*\int \frac{dx}{x\sqrt{1-(\ln x)^2}} \quad , u = \ln x \to du = \frac{1}{x} \cdot dx \to dx = x \cdot du$

$\Rightarrow \int \frac{\cancel{x} \cdot du}{\cancel{x}\sqrt{1-u^2}}$

$= \sin^{-1}(u) + c$

$= \sin^{-1}(\ln x) + c$ #

$*\int \frac{dx}{\sqrt{x}\,(1+x)} \quad , u = \sqrt{x} \to du = \frac{1}{2\sqrt{x}} \cdot dx \to dx = 2\sqrt{x} \cdot du$

$x = u^2$

$= \int \frac{2\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}\,(1+u^2)} \cdot du$

$= 2\tan^{-1}(u) + c$

$= 2\tan^{-1}(\sqrt{x}) + c$

lecture 2: Integration by parts

6/Mar/2022

First

$\triangle\ (f(x)\cdot g(x))' = f(x)\cdot g'(x) + g(x)\cdot f'(x)$

$\int (f(x)\cdot g(x))'\cdot dx = \int f(x)\cdot g'(x)\cdot dx + \int g(x)\cdot f'(x)\cdot dx.$

$f(x)\cdot g(x) = \int f(x)\cdot g'(x)\cdot dx + \int g(x)\cdot f'(x)\cdot dx$

(قانون الاجزاء)

$\therefore \int f(x)\cdot g'(x)\cdot dx = f(x)\cdot g(x) - \int g(x)\cdot f'(x)\cdot dx.$ #

$u = f(x) \Rightarrow \frac{du}{dx} = f'(x)$

$v = g(x) \Rightarrow \frac{dv}{dx} = g'(x)$

$\therefore \int u\cdot dv = u\cdot v - \int v\cdot du$

$$\begin{array}{ccc} u & \oplus & dv \\ \downarrow & \searrow & \downarrow \\ du & \xrightarrow{-\int} & v \end{array}$$

* $\int x\ln x\cdot dx.$

$$\begin{array}{ccc} u & & dv \\ \ln x & & x \\ \downarrow & \ominus\searrow & \downarrow \\ \frac{1}{x} & \xrightarrow{-\int} & \frac{x^2}{2} \end{array}$$

$\therefore I = \frac{x^2}{2}\ln x - \int \frac{x}{2}\cdot dx$

$I = \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + C$

* $\int (\ln x)^2\cdot dx$

$$\begin{array}{ccc} u & & dv \\ (\ln x)^2 & & 1 \\ \downarrow & \oplus\searrow & \downarrow \\ \frac{2\ln x}{x} & \xrightarrow{-\int} & x \end{array}$$

$\therefore I = x(\ln x)^2 - \int 2\ln x$

$I = x(\ln x)^2 - 2[x\ln x - x] + C$

• note that: $(\ln x)^n \neq \ln x^n$

$\int (\ln x)^n\cdot dx = n\int \ln x\cdot dx$

* $\int \sin^{-1}x\cdot dx$

$$\begin{array}{ccc} u & & dv \\ \sin^{-1}x & & 1 \\ \downarrow & \oplus\searrow & \\ \frac{1}{\sqrt{1-x^2}} & \xrightarrow{-\int} & x \end{array}$$

• $u = 1-x^2$

$du = -2x\cdot dx$

$dx = \frac{du}{-2x}$

$\therefore I = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}$

$= x\sin^{-1}x - \int \frac{x}{\sqrt{u}}\cdot\frac{du}{-2x}$

$= x\sin^{-1}x + \frac{1}{2}\int u^{-\frac{1}{2}}\cdot du$

$= x\sin^{-1}x + \frac{1}{2}\cdot 2u^{\frac{1}{2}} + C$

$= x\sin^{-1}x + \sqrt{1-x^2} + C$

lecture 2: Integration by parts 6/Mar/2022

#First

* $\int x \tan^{-1}x \, dx$

u		dv
$\tan^{-1}x$	+	x
$\frac{1}{1+x^2}$	$-\int$	$\frac{x^2}{2} \, dx$

$$\therefore I = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2} \, dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2+1}{1+x^2} - \int \frac{1}{1+x^2}$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[x - \tan^{-1}x\right] + C \ \#$$

* $\int x^2 e^x \, dx$, u dv

u		dv
x^2	+	e^x
$2x$	−	e^x
2	+	e^x
0		e^x

$$\therefore I = x^2e^x - 2xe^x + 2e^x + C$$

* $\int 2x^3 \cos x \, dx$, u dv

u		dv
$2x^3$	+	$\cos x$
$6x^2$	−	$\sin x$
$12x$	+	$-\cos x$
12	−	$-\sin x$
0		$\cos x$

$$\therefore I = 2x^3 \sin x + 6x^2 \cos x - 12x \sin x - 12 \cos x + C \ \#$$

* $\int x \sec^2 x \tan x \, dx$, u dv

u		dv
x	+	$\sec^2 x \tan x$
1	−	$\frac{\tan^2 x}{2}$
0		$\frac{1}{2}\tan x - \frac{x}{2}$

$$\therefore I = \frac{1}{2}x\tan^2 x + \frac{x}{2} - \frac{1}{2}\tan x + C \ \#$$

- $\int \sec^2 x \tan x \, dx$ $u = \tan x$, $du = \sec^2 x \, dx$, $dx = \frac{du}{\sec^2 x}$

$$= \int \cancel{\sec^2 x} \cdot u \cdot \frac{du}{\cancel{\sec^2 x}}$$

$$= \frac{u^2}{2} + C$$

- $\frac{1}{2}\int \tan^2 x \, dx \rightarrow \frac{1}{2}\int \sec^2 x - 1$

$$\frac{1}{2}(\tan x - x)$$

lecture 2: Integration by parts   6/Mar/2022

First

أولاً: تعويض

ثانياً: أجزاء

$$*\int e^{\sqrt{x}}\,dx \quad , \text{let } u=\sqrt{x}$$

$$du = \frac{1}{2\sqrt{x}}\,dx$$

$$dx = 2\sqrt{x}\,du$$

$$= \int e^{u}\cdot 2u\,du$$

u		dv
$2u$	$+$	e^{u}
2	$-$	e^{u}
0		e^{u}

$$\therefore I = 2ue^{u} - 2e^{u} + c$$

$$= 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + c$$

$$*\int e^{x}\cdot \sin x\,dx$$

u		dv
e^{x}	$+$	$\sin x$
e^{x}	$-$	$-\cos x$
e^{x}	$+\int$	$-\sin x$

عند تكرار دوال موقف

$$\Rightarrow \int e^{x}\cdot \sin x\,dx = -e^{x}\cos x + e^{x}\sin x - \int e^{x}\sin x\,dx$$

$$\cancel{2}\int e^{x}\cdot \sin x\,dx = \frac{-e^{x}\cos x}{2} + \frac{e^{x}\sin x}{2} + c$$

$$*\int \sin(\ln x)\,dx \quad , \text{let } u=\ln x$$

$$du = \frac{1}{x}\,dx$$

$$dx = x\,du$$

$$e^{u} = x$$

$$\int \sin u\cdot x\,du$$

$$\int e^{u}\cdot \sin u\,du$$

نفس السؤال السابق

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lecture 3: Integration of rational functions by partial fractions. 7/Mar/2022

#First

* $\int \frac{poly}{poly} = \int \frac{f(x)}{g(x)}$, case 1 : If deg $f(x) \geq$ deg $g(x) \Longrightarrow$ long division.
قسمة طويلة

* $\int \frac{x^3 + 2x}{x^2 - 1} \cdot dx$

$$\begin{array}{r|l} & x^2 + x + 3 \\ \hline x - 1 & x^3 + 2x \\ & \ominus x^3 \oplus x^2 \\ \hline & x^2 + 2x \\ & \ominus x^2 \oplus x \\ \hline & 3x \\ & 3x \oplus 3 \\ \hline & (3) \end{array}$$

$= \int$ ناتج $+ \int \frac{\text{باقي}}{\text{المقسوم عليه}}$

$= \int x^2 + x + 3 + \int \frac{3}{x-1}$

$= \frac{x^3}{3} + \frac{x^2}{2} + 3x + 3\ln|x-1| + c$ #

Case 2 : If deg $f(x) <$ deg $g(x)$

اول شي منفكر فيه

↳ 1-) $\int \frac{f'(x)}{f(x)} = \ln|f(x)| + c$

* $\int \frac{x^2}{x^3+1} \cdot dx = \frac{1}{3} \int \frac{3x^2}{x^3+1} \cdot dx = \frac{1}{3} \ln|x^3+1| + c$

2) $\int \frac{f(x)}{g(x)} = \int \frac{f(x)}{(a_1x+b_1)(a_2x+b_2)(a_3x+b_3)\ldots}$

$= \int \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2} + \frac{C}{a_3x+b_3} + \ldots\ldots$

* $\int \frac{5}{x^2-1} \cdot dx = \int \frac{5}{(x-1)(x+1)} \cdot dx = \int \frac{A}{x-1} + \frac{B}{x+1} \cdot dx$

Now. $5 = A(x+1) + B(x-1)$

at $x = -1 \rightarrow \frac{5}{2} = \frac{-2B}{-2} \rightarrow B = -5/2$

at $x = +1 \rightarrow \frac{5}{2} = \frac{2A}{2} \rightarrow A = 5/2$

$\Longrightarrow \int \frac{5/2}{x-1} + \frac{-5/2}{x+1} \cdot dx$

$= 5/2 \ln|x-1| - 5/2 \ln|x+1|$

$= \ln|x-1|^{5/2} - \ln|x+1|^{5/2}$

$= \ln\left(\frac{x-1}{x+1}\right)^{5/2}$

lecture 3: 7/Mar/2022

First

3.) $\displaystyle\int \frac{f(x)}{(a_1x+b_1)^2(a_2x+b_2)^3 \dots}$

مكرر

$$= \int \frac{A}{(a_1x+b_1)} + \frac{B}{(a_1x+b_1)^2} + \frac{C}{(a_2x+b_2)} + \frac{D}{(a_2x+b_2)^2} + \frac{E}{(a_2x+b_2)^3}$$

* find the form of the partial fraction decomposition of:

1. $\displaystyle\frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$

2. $\displaystyle\frac{x+1}{x^3+3x^2} = \frac{x+1}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$

3. $\displaystyle\frac{3}{(x^2+2x+1)^2(x-1)^3} = \frac{3}{((x+1)^2)^2(x-1)^3} = \frac{3}{(x+1)^4(x-1)^3}$

$$= \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x+1)^4} + \frac{E}{(x-1)} + \frac{F}{(x-1)^2} + \frac{G}{(x-1)^3}$$

4.) $\displaystyle\int \frac{f(x)}{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\dots}$

لا غير قابل للتحليل   لا غير قابل للتحليل

$$= \int \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \frac{A_2x+B_2}{a_2x^2+b_2x+c_2}$$

5.) $\displaystyle\int \frac{f(x)}{(a_1x^2+b_1x+c_1)^2(a_2x^2+b_2x+c_2)^3 \dots}$

لا غير قابل للحل

$$= \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \frac{A_2x+B_2}{(a_1x^2+b_1x+c_1)^2} + \frac{A_3x+B_3}{(a_2x^2+b_2x+c_2)} + \frac{A_4x+B_4}{(a_2x^2+b_2x+c_2)^2} + \frac{A_5x+B_5}{(a_2x^2+b_2x+c_2)^3} + \dots$$

lecture 3: 7/Mar/2022

#First

$* \int \frac{\cos x}{\sin^3 x + \sin x} \cdot dx$, let $u = \sin x$

$du = \cos x \cdot dx$

$dx = \frac{du}{\cos x}$

$= \int \frac{\cancel{\cos x}}{u^3+u} \cdot \frac{du}{\cancel{\cos x}}$

$= \int \frac{1}{u^3+u} \cdot du$

$= \int \frac{1}{\underbrace{u(u^2+1)}_{\text{كلياً}}} = \int \frac{A}{u} + \frac{Bu+C}{u^2+1}$

$$1 = A(u^2+1) + (Bu+C)u$$

$* u=0 \Rightarrow A=1$

$* u=1 \Rightarrow 1 = 2A + B + C \Rightarrow 1 = 2 + B + C \Rightarrow B+C=-1$

$* u=-1 \Rightarrow 1 = 2A - C + B \Rightarrow 1 = 2 + B - C \Rightarrow B-C=-1$

$2B = -2$

$B=-1$ $\quad C=0$

$\Rightarrow \int \frac{1}{u} - \frac{u}{u^2+1} \cdot du$

$= \ln|u| - \frac{1}{2}\ln|u^2+1| + c$

$= \ln|\sin x| - \frac{1}{2}\ln|\sin^2 x + 1| + c$ #

$* \int \frac{2x+4}{x^3-2x^2} \cdot dx = \int \frac{2x+4}{\underbrace{x^2(x-2)}_{\text{كلياً}}} \cdot dx = \int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \cdot dx$

$$2x+4 = Ax(x-2) + B(x-2) + C(x^2)$$

- at $x=0 \rightarrow 4 = -2B \rightarrow B=-2$
- at $x=2 \rightarrow 8 = 4C \rightarrow C=2$
- at $x=1 \rightarrow 6 = -A + 2 + 2 \rightarrow A=-2$

ليش عنده بلا حاجة بس ماخذ احنا

$\therefore I = \int \frac{-2}{x} - \frac{2}{x^2} + \frac{2}{x-2}$

$= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + c$

$* \int \frac{-2}{x^2} \cdot dx$

$\int -2x^{-2} \cdot dx$

$-2\frac{x^{-1}}{-1}$

$= \frac{2}{x}$

lecture 3: 8/Mar/2022

First

* Find the form of the partial fraction decomposition of:

1) $\frac{x+1}{x^3+4x} = \frac{x+1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

2) $\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

3) $\frac{x^2+1}{x^3(x+1)(x^2+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+J}{(x^2+1)^2}$

4) $\frac{x+5}{(x^2+x+1)^2(x-2)(x^2-1)^2} = \frac{Ax+B}{(x^2+x+1)} + \frac{Cx+d}{(x^2+x+1)^2} + \frac{E}{(x-2)} + \frac{F}{(x-1)} + \frac{G}{(x-1)^2} + \frac{H}{(x+1)} + \frac{I}{(x+1)^2}$

لا يتحلل $\quad ((x-1)(x+1))^2$

5) $\frac{1}{x^4+4x^2+3} = \frac{1}{(x^2+3)(x^2+1)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2+1}$

Identities:

* $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
* $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
* $\sin(2x) = 2\sin x \cos x$
* $\cos(2x) = \cos^2 x - \sin^2 x$
  $= 2\cos^2 x - 1$
  $= 1 - 2\sin^2 x$
* $\sin^2 x + \cos^2 x = 1$
* $\tan^2 + 1 = \sec^2 x$
* $1 + \cot^2 x = \csc^2 x$
* $\cos(-x) = \cos x$
* $\sin(-x) = -\sin x$
* $\sin^2 x = \frac{1-\cos 2x}{2}$
* $\cos^2 x = \frac{1+\cos 2x}{2}$

lecture4: Trigonometric Integrals. (part1)   9 / Mar / 2022

First

- $\int \sin^n x \, dx = \frac{-1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} . dx$

- $\int \cos^n x . dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x . dx$

- to solve $\int \sin^m x \cos^n x . dx$:-

△1 if n odd: بنعمل هنا على طول مش زي الاس الفردي

1.) Split off a factor of $\cos x$
2.) apply the identity $\cos^2 x = 1 - \sin^2 x$
3.) make the substitution $u = \cos x$

△2 if m odd:

1.) split off a factor of $\sin x$
2.) apply the identity $\sin^2 x = 1 - \cos^2 x$
3.) make the substitution $u = \cos x$

△3 If m and n even: Use the identities $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

لو بنحل بالطريقة العادية،

$* \int \sin^2 x . dx = \int \frac{1}{2}(1 - \cos 2x) . dx$

$= \frac{1}{2}(x - \frac{1}{2} \sin 2x) + c$

على النظرية الجديدة

راس cos

$* \int \cos^2 x . dx = \frac{1}{2} \cos^1 x \sin x + \frac{1}{2} \int 1 \, dx = \frac{1}{4} \sin 2x + \frac{1}{2} x + c$

$\frac{1}{2} \times \sin 2x = 2(\sin x \cos x) \frac{1}{2}$

$* \int \cos^3 (2x) . dx$, let $u = 2x$

$du = 2dx$

$\int \frac{1}{2} \cos^3(u) . du = \frac{1}{2}\left(\frac{1}{3} \cos^2(u) \sin u + \frac{2}{3} \int \cos(u) . du\right)$

$= \frac{1}{2}\left(\frac{1}{3} \cos^2(2x) \sin 2x + \frac{2}{3} \sin 2x\right) + c$

lecture 4:
#first

9/Mar/2022

$$* \int \sin^5 x \, dx = \frac{-1}{5}\sin^4 x\cos x + \frac{4}{5}\int \sin^3 x \, dx$$

$$= \frac{-1}{5}\sin^4 x\cos x + \frac{4}{5}\left(\frac{-1}{3}\sin^2 x\cos x + \frac{2}{3}\int \sin x \, dx\right)$$

$$= \frac{-1}{5}\sin^4 x\cos x + \frac{4}{5}\left(\frac{-1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x\right) + c$$

$$* \int \sin^3 x\sqrt{\cos x} \, dx$$

let $u = \cos x$
$du = -\sin x \, dx$

$$\int \sin^{\cancel{3}\,2} x\sqrt{u} \cdot \frac{du}{-\cancel{\sin x}}$$

$$-\int (1-\cos^2 x)\sqrt{u} \, du$$

$$-\int (1-u^2)(\sqrt{u}) \, du$$

$$= -\left(\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{7}u^{\frac{7}{2}}\right) + c = -\left(\frac{2}{3}(\cos x)^{3/2} - \frac{2}{7}(\cos x)^{7/2}\right) + c$$

$$* \int \cos^4 x \sin x \, dx$$, let $u = \cos x$
$du = -\sin x \, dx$

$$= \int u^4 \cdot \cancel{\sin x} \cdot \frac{du}{-\cancel{\sin x}}$$

$$= \frac{-u^5}{5} + c = \frac{-\cos^5 x}{5} + c$$

$$* \int \sin^{3/2} x \cos^3 x \, dx = \int \sin^{3/2} x \cdot \cos^2 x \cdot \cos x \, dx$$

$$= \int \sin^{3/2} x(1-\sin^2 x) \cdot \cos x \, dx$$, let $u = \sin x$
$du = \cos x \, dx$

$$= \int u^{3/2} - u^{7/2} \cdot \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$$

$$= \frac{2}{5}u^{5/2} - \frac{2}{9}u^{9/2} + c$$

$$= \frac{2}{5}\sin^{5/2} x - \frac{2}{9}\sin^{9/2} x + c$$

$$* \int \sin^2(2x)\cos^3(2x) \, dx$$

$$= \int \sin^2(2x) \cdot \cos^2(2x) \cdot \cos 2x \, dx$$

$$= \int \sin^2(2x)(1-\sin^2(2x)) \cdot \cos 2x \, dx$$, let $u = \sin 2x$
$du = 2\cos 2x \, dx$

$$= \int u^2 - u^4 \cdot \cancel{\cos 2x} \cdot \frac{du}{2\cancel{\cos 2x}}$$

$$= \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + c = \frac{1}{2}\left(\frac{\sin^3 2x}{3} - \frac{\sin^5 2x}{5}\right) + c$$

Lecture 4: 9/Mar/2022

First

$\int \sin^{3} x \cos^4 x \, dx$

$\int \sin x(\sin^2 x) \cdot \cos^4 x \, dx$

$\int \sin x(1-\cos^2 x) \cdot \cos^4 x \, dx$, let $u = \cos x$
$du = -\sin x \, dx$

$\int \cancel{\sin x} \cdot u^4 - u^6 \cdot \frac{du}{-\cancel{\sin x}}$

$-\frac{u^7}{7} - \frac{u^5}{5} + c = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + c$

* $\int \sin^3 x \cdot \cos^7 x \, dx = \int \sin x(1-\cos^2 x) \cdot \cos^7 x \, dx$, let $u = \cos x$
$du = -\sin x \, dx$

ايبراو نمسحه بالربط

$= \int \cancel{\sin x}(u^7 - u^9) \cdot \frac{du}{-\cancel{\sin x}}$

$= \frac{u^{10}}{10} - \frac{u^8}{8} + c$

$= \frac{\cos^{10} x}{10} - \frac{\cos^8 x}{8} + c$

* $\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2}(1-\cos 2x) \cdot \frac{1}{2}(1+\cos 2x) \, dx$

$= \frac{1}{4}\int (1-\cos^2 2x)$

$= \frac{1}{4}\int (1 - (\frac{1}{2}(1+\cos 4x)) \, dx$

⋮

or... $\int (\sin x \cos x)^2 \, dx = \int (\frac{1}{2}\sin 2x)^2 \, dx$

$= \frac{1}{4}\int \frac{1}{2}(1-\cos 4x) = \frac{1}{8}(x - \frac{1}{4}\sin 4x) + c$

* $\int \sqrt{\sin x} \cdot \cos^5 x \, dx$, $u = \sin x$
$du = \cos x \, dx$

$= \int \sqrt{u} \cdot (1-u^2)^2 \cdot \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$

$= \int \sqrt{u}(1 - 2u^2 + u^4) \, du$

$\int (\sqrt{u} - 2\sqrt{u}\,u^2 + \sqrt{u} \cdot u^4) \, du$

$\frac{2}{3}u^{3/2} - \frac{4}{7}u^{7/2} + \frac{2}{11}u^{11/2} + c$

$\frac{2}{3}(\sin x)^{3/2} - \frac{4}{7}(\sin x)^{7/2} + \frac{2}{11}(\sin x)^{11/2} + c$

* H.W: 1-) $\int \sin^7 x \, dx$ $\quad (\sin^2 x)^3$

$= \int \sin x(\sin^6 x) \, dx$

$= \int \sin x(1-\cos^2 x)^3 \, dx$. let $u = \cos x$
$du = -\sin x \, dx$

$= \int \cancel{\sin x}(1-u^2)^3 \cdot \frac{du}{-\cancel{\sin x}}$

$= -\int 1 - 2u^2 + u^4(1-u^2) \, du$

$= -\int (1-u^2) - 2u^2 + 2u^4 + u^4 - u^6 \, du$

$= -\int 1 - 3u^2 + 3u^4 - u^6 \, du$

$= -u + u^3 - \frac{3}{5}u^5 + \frac{u^7}{7} + c$

$= -\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{\cos^7 x}{7} + c$

Lecture 4:
#First

9/Mar/2022

H.W

2-) $\int \sin^3(2x)\cos^3(2x)\,.dx$

$\int \sin(2x)(1-\cos^2 2x)\cdot(\cos^3(2x))$, let $u=\cos 2x$

$du = -2\sin 2x\,.dx$

$= \int \cancel{\sin 2x}\cdot(u^3-u^5)\cdot\frac{du}{-2\cancel{\sin 2x}}$

$= -\frac{1}{2}\left(\frac{u^4}{4}-\frac{u^6}{6}\right)+c$

$= -\frac{1}{2}\left(\frac{\cos^4 2x}{4}-\frac{\cos^6 2x}{6}\right)+c$

$* \int \sin\frac{x}{2}\cos x\,.dx$

$= \frac{1}{2}\int \sin\frac{-x}{2} + \sin\frac{3}{2}x\,.dx$

$= \frac{1}{2}\left(2\cos\frac{-x}{2} - \frac{2}{3}\cos\frac{3}{2}x\right)+c$

[illegible]

3-) $\int \sin^2 x\cos^4 x\,.dx$

$= \int (1-\cos^2 x)\cos^4 x\,.dx$

$= \int \cos^4 x - \cos^6 x\,.dx$

$\int \left(\frac{1}{2}(1+\cos 2x)\right)^2 - \frac{1}{2}(1+\cos 2x)\,.dx$

$\int \frac{1}{4}(1+\cos 2x)^2 - \frac{1}{2}(1+\cos 2x)\,.dx$

$\int \frac{1}{4}(1+2\cos 2x+\cos^2 2x) - \frac{1}{2}(1+\cos 2x)\,.dx$

$\frac{1}{4}\left(1+2\cos 2x+\frac{1}{2}(1+\cos 4x)\right) - \frac{1}{2}(1+\cos 2x)\,.dx$

$= \frac{1}{4}(x+\sin 2x)+\frac{1}{2}\cdot\frac{1}{4}\left(x+\frac{1}{4}\sin 4x\right) - \frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)+c$

How to integrate:-

$\int \sin(ax)\cos(Bx)\,.dx = \frac{1}{2}\int (\sin(ax-Bx)+\sin(ax+Bx))\,.dx$

$\int \sin(ax)\sin(Bx)\,dx = \frac{1}{2}\int (\cos(ax-Bx)-\cos(ax+Bx))\,.dx$

$\int \cos(ax)\cos(Bx)\,.dx = \frac{1}{2}\int (\cos(ax-Bx)+\cos(ax+Bx))\,.dx$

$\int \cos(ax)\sin(Bx)\,.dx = \frac{1}{2}\int (\sin(ax+Bx)-\sin(ax-Bx))\,.dx$

$* \int \sin(7x)\cos(3x)\,.dx$

$= \frac{1}{2}\int (\sin 4x+\sin 10x)\,.dx = \frac{1}{2}\left(\frac{-1}{4}\cos 4x - \frac{1}{10}\cos 10x\right)+c$

lecture 5:
#First

11 / Mar / 2022

- $\int \tan^n x \, dx = \frac{\tan^{n-1}}{n-1} - \int \tan^{n-2} x \, dx$

- $\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$

- to integrate $\int \tan^m x \sec^n x \, dx$

1) if n even:
① split off a factor of $\sec^2 x$
② apply the identity $\sec^2 x = \tan^2 x + 1$
③ make the substitution $u = \tan x$

2) if m odd:
① split off a factor of $\sec x \tan x$
② apply the identity $\tan^2 x = \sec^2 x - 1$
③ make the substitution $u = \sec x$

3) if m even, n odd:
Use the identities (متطابقات) to reduce the integral to powers of $\sec x$ alone, then use the reduction formula (تكرار تخفيض) for powers of $\sec x$.

- To integrate $\int \cot^m x \csc^m x \, dx$, use $\cot^2 x = \csc^2 x - 1$

- $\int \tan x \, dx = -\ln|\cos x| + c$
- $\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x + c$
- $\int \sec^2 x \, dx = \tan x + c$
- $\int \sec x \, dx = \ln|\sec x + \tan x| + c$ احفظها

$\int \sec^3 x \, dx = \int \sec^2 x \cdot \sec x \, dx$, $u = \sec x$, $dv = \sec^2 x$; $du = \sec x \tan x$, $v = \tan x$

$\Rightarrow = \sec x \tan x - \int \sec x \tan^2 x \, dx$

$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$

$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x + \int \sec x \, dx$

$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$

$\int \sec^3 x \, dx = \frac{1}{2} (\sec x \tan x + \ln|\sec x + \tan x|) + c$

Five Apple

lectures:

#find

11/Mar/2022

#

6) $\int \tan^4(\frac{x}{2}).dx$, $u=\frac{x}{2}$

$= 2\int \tan^4 u \, du = 2\left(\frac{\tan^3 u}{3} - \int \overset{\sec^2-1}{\tan^2 u}\right) = 2\left(\frac{\tan^3(\frac{x}{2})}{3} - \tan(\frac{x}{2}) + \frac{x}{2}\right) + c$

7) $\int_0^{\pi/4} \sec^3 x \, dx = \frac{\sec x \tan x}{2}\Big]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec x \, dx$

$= \frac{\sqrt{2}}{2} + \frac{1}{2}\left(\ln|\sec x + \tan x|\Big]_0^{\pi/4}\right)$

$= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1)$ #

8) $\int \sec^4(2x)\tan^6(2x)\,dx$, $u=2x$, $du=2dx$

$= \frac{1}{2}\int \sec^4(u)\tan^6(u).du$

$= \frac{1}{2}\int \sec^2 u(\sec^2(u))\tan^6(u).du$

$= \frac{1}{2}\int \sec^2(u)(\tan^2 u+1)\tan^6(u).du$, $v=\tan u$, $dv = \sec^2 u.du$

$= \frac{1}{2}\int \cancel{\sec^2(u)}(v^2+1)v^6 \frac{dv}{\cancel{\sec^2 u}}$

$= \frac{1}{2}\left(\frac{v^9}{9}+\frac{v^7}{7}\right)+c = \frac{1}{2}\left(\frac{\tan^9(2x)}{9}+\frac{\tan^7(2x)}{7}\right)+c$

9) $\tan^5 x \cdot \sec^4 x \, dx$

$\int \tan^5 x(\tan^2 x+1)\sec^2 x \, dx$, $u=\tan x$ or $\int \tan x \sec x(\sec^2 x-1)\sec^3 x\,dx$, $u=\sec x$

10) $\int \sqrt{\tan x}\,\sec^4 x \, dx$, $u=\tan x$, $du=\sec^2 x\,dx$

$\int \sqrt{u}\,(u^2+1)\,\cancel{\sec^2 x}.\frac{du}{\cancel{\sec^2 x}}$

11) $\int \cot^3 x \csc x \, dx$

$= \int \cot^2 x \cot x \csc x.dx$

$= \int(\csc^2 x-1)\cot x.\csc x$, $u=\csc x \Rightarrow du=-\csc x\cot x\,dx$

~~$= \int u^3 - u . \cot x$~~

$= -\int \cot x\csc^{3}x - \cot x\csc x.\frac{du}{\csc x\cot x}$

$= -\int u^2-1\,du$

$= -\left(\frac{u^3}{3}-u\right)+c$

Five Apple

lecture 5:
First

11 / Mar / 2022

12-) $\int \tan^2 x \sec x \, dx$

$= \int (\sec^2 x - 1) \sec x \, dx$

$= \int \sec^3 x - \sec x \, dx$

$= \dfrac{\sec x \tan x}{2} + \dfrac{1}{2} \ln|\sec x + \tan x| - \ln|\sec x + \tan x| + c.$

$= \dfrac{\sec x \tan x}{2} - \dfrac{1}{2} \ln|\sec x + \tan x| + c$

* lecture 6: Trigonometric substitutions..

expression:	Trig sub:	θ
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$0 \leq \theta \leq \frac{\pi}{2}$ if $x \geq a$ $\frac{\pi}{2} \leq \theta \leq \pi$ if $x \leq a$

• note that: $\sqrt{a^2 - x^2}$, $x = a\sin\theta$

$\sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)}$

$= \sqrt{a^2}\sqrt{\cos^2\theta}$

$= a\cos\theta$ since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta}$

$= \sqrt{a^2}\left(\sqrt{1 + \tan^2\theta}\right)$

$= a\sec\theta$

• $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)}$

$= a\sqrt{\tan^2\theta}$

$= a\tan\theta$

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lecture 6:
#First

12/Mar/2022

هون بكون بدو شو التعويض المناسب؟

* write the trig substituation:

$a = \sqrt{4}$

1.) $\int \sqrt{4+x^2} \rightarrow x = 2\tan\theta$

$\sqrt{a^2 - b^2x^2} \rightarrow bx = a\sin\theta$

2.) $\int \sqrt{x^2-5} \rightarrow x = \sqrt{5}\sec\theta$

$\sqrt{a^2 + b^2x^2} \rightarrow bx = a\tan\theta$

3.) $\int \sqrt{1+(x+2)^2} \rightarrow x+2 = 1\tan\theta$

$\sqrt{b^2x^2 - a^2} \rightarrow bx = a\sec\theta$

4.) $\int \frac{dx}{(9-x^2)^{3/2}} \rightarrow x = 3\sin\theta$

5.) $\int \frac{dx}{(2x^2-3)^{3/2}} \rightarrow \sqrt{2}x = \sqrt{3}\sec\theta$

* $\int \frac{\sqrt{x^2-16}}{2x} \, dx$

$\boxed{x = 4\sec\theta} \Longrightarrow \boxed{dx = 4\sec\theta\tan\theta\, d\theta}$

$\boxed{\sec\theta = \frac{x}{4}}$

$\therefore I = \int \frac{\sqrt{16\sec^2\theta - 16}}{2(4\sec\theta)} \cdot 4\sec\theta\tan\theta\, d\theta$

$I = \int \frac{\cancel{4}^{2}\sqrt{\sec^2\theta - 1}}{\cancel{2}} \cdot \tan\theta \, d\theta$

$= 2\int \tan^2\theta \, d\theta$

~~$= 2\sec\theta \ldots$~~

$= 2\int \sec^2\theta - 1 \, d\theta$

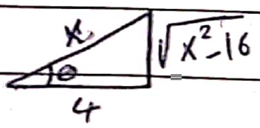
$= 2(\tan\theta - \theta) + C$

ارجع للفرض لنعبر θ

$\sec\theta = \frac{x}{4}$

$\sec^{-1}(\sec\theta) = \sec^{-1}\frac{x}{4}$

$\theta = \sec^{-1}\frac{x}{4}$



مثلث

$\Rightarrow \therefore I = 2(\tan\theta - \theta) + C$

$= 2\left(\frac{\sqrt{x^2-16}}{x} - \sec^{-1}\frac{x}{4}\right) + C$

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lecture 6: ..

#First

12/Mar/2022

$$*\int \frac{\sqrt{x^2+1}}{x}\,dx$$

$$\boxed{x=\tan\theta} \Rightarrow \boxed{dx=\sec^2\theta\,d\theta}$$

$$=\int \frac{\sqrt{\tan^2\theta+1}}{\tan\theta}\cdot \sec^2\theta\,d\theta$$

$$=\int \frac{\sec\theta\cdot(1+\tan^2\theta)}{\tan\theta}\,d\theta$$

$$=\int \frac{\sec\theta+\sec\theta\tan^2\theta}{\tan\theta}\,d\theta$$

$$=\int \frac{\sec\theta}{\tan\theta}\,d\theta+\int \frac{\sec\tan^2\theta}{\tan\theta}\,d\theta$$

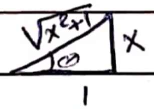
$$=\int \frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}\,d\theta+\int \sec\theta\tan\theta\,d\theta$$

$$=\int \csc\theta\,d\theta+\int \sec\theta\tan\theta\,d\theta$$

$\int \sec$

$$=-\ln|\csc\theta+\cot\theta|+\sec\theta+C$$

Now, $\frac{x}{1}=\tan\theta$



$$\therefore I=-\ln|\csc\theta+\cot\theta|+\sec\theta+c$$

$$=-\ln\left|\frac{\sqrt{x^2+1}}{x}+\frac{1}{x}\right|+\frac{\sqrt{x^2+1}}{1}+c$$

$$*\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$$

$x=2\sin\theta \rightarrow dx=2\cos\theta\,d\theta$

$\theta=\sin^{-1}\left(\frac{x}{2}\right)$, if $x=1 \Rightarrow \theta=\sin^{-1}\frac{1}{2}=\frac{\pi}{6}$

, if $x=\sqrt{2} \Rightarrow \theta=\sin^{-1}\frac{\sqrt{2}}{2}=\frac{\pi}{4}$

$$=\int_{\pi/6}^{\pi/4} \frac{2\cos\theta\,d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}$$

$$=\int_{\pi/6}^{\pi/4} \frac{2\cos\theta}{4\sin^2\theta(4-4\sin^2\theta)}$$

$$=\int_{\pi/6}^{\pi/4} \frac{2\cos\theta\,d\theta}{4\sin^2\theta\sqrt{4}(\sqrt{1-\sin^2\theta})}$$

$$=\int_{\pi/6}^{\pi/4} \frac{\cos\theta\,d\theta}{4\sin^2\theta\cdot\cos\theta}$$

$$=\frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2\theta\,d\theta$$

$$=\frac{1}{4}\left(-\cot\theta\Big]_{\pi/6}^{\pi/4}\right)$$

$$=\frac{1}{4}\left(-\cot\frac{\pi}{4}+\cot\frac{\pi}{6}\right)$$

$$=\frac{1}{4}\left(-1+\sqrt{3}\right)$$

lecture 6:

#First

12/Mar/2022

$$* \int \frac{dx}{\sqrt{5-4x-2x^2}}$$

⚠ شرط اكمال المربع

معامل x^2 موجب + بنضيف ونطرح $\left(\frac{b}{2}\right)^2$

$$= \int \frac{dx}{\sqrt{-2\left(x^2+2x-\frac{5}{2}\right)}}$$

$$= \int \frac{dx}{\sqrt{-2\left(x^2+2x+1-1-\frac{5}{2}\right)}}$$

$$= \int \frac{dx}{\sqrt{-2\left((x+1)^2-\frac{7}{2}\right)}} = \int \frac{dx}{\sqrt{7-2(x+1)^2}}$$

, let $\sqrt{2}(x+1) = \sqrt{7}\sin\theta$

$x+1 = \frac{\sqrt{7}}{\sqrt{2}}\sin\theta$

$dx = \frac{\sqrt{7}}{\sqrt{2}}\cos\theta \, d\theta$

$$= \int \frac{\sqrt{7}\cos\theta \, d\theta}{\sqrt{2}\sqrt{7-2\left(\frac{7}{2}\sin^2\theta\right)}}$$

$$= \int \frac{\cancel{\sqrt{7}}\cos\theta \, d\theta}{\sqrt{2}\,\cancel{\sqrt{7}}\left(\sqrt{1-\sin^2\theta}\right)}$$

$$= \int \frac{1}{\sqrt{2}} \cdot \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}} \, d\theta$$

$$= \frac{1}{\sqrt{2}} \int d\theta = \frac{1}{\sqrt{2}}\theta + C$$

Now, $\sqrt{2}(x+1) = \sqrt{7}\sin\theta$

$\frac{\sqrt{2}}{\sqrt{7}}(x+1) = \sin\theta$

$\theta = \sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{7}}(x+1)\right)$

$$\therefore I = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{7}}(x+1)\right) + C$$

lecture 6:

\# First

12 / Mar / 2022

$\int d\theta = \sec^{-1}(2\sin x)$

$* \int \frac{x}{\sqrt{x^2-4x+8}} \cdot dx$

غير قابل للتحليل

لازم

$= \int \frac{x}{\sqrt{x^2-4x+4-4+8}}$

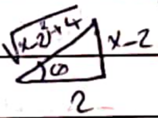
$= \int \frac{x}{\sqrt{(x-2)^2+4}}\, dx$, let $x-2 = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$= \int \frac{2\tan\theta+2}{\cancel{4}\tan^2+4} \cdot 2\sec^2\theta\, d\theta$

$= \int \frac{\cancel{4}(\tan\theta+1)\cancel{\sec^2\theta}}{\cancel{4}(\cancel{\tan^2\theta+1})} \cdot d\theta$

$= -\ln|\cos\theta| + \theta + C$, Now $\tan\theta = \frac{x-2}{2}$



$= -\ln\left|\frac{2}{\sqrt{(x-2)^2+4}}\right| + \tan^{-1}\left(\frac{x-2}{2}\right) + c$

$* \int \frac{\cos x}{\sin x\sqrt{4\sin^2 x-1}}\, dx$

let $u = \sin x$

$du = \cos x\, dx$

$\int \frac{\cancel{\cos x}}{u\sqrt{4u^2-1}} \cdot \frac{du}{\cancel{\cos x}}$

$= \int \frac{1}{u\sqrt{4u^2-1}} \cdot du$

let $2u = 1\sec\theta$

$2du = \sec\theta\tan\theta\, d\theta$

$= \int \frac{\sec\theta\tan\theta}{2\left(\frac{\sec\theta}{2}\right)\sqrt{4\frac{\sec^2}{4}-1}} \cdot d\theta$

$= \int \frac{\cancel{\sec\theta}\cancel{\tan\theta}}{\cancel{\sec\theta}\cancel{\tan\theta}} \cdot d\theta$

$= \theta + C$

Now,, $2u = \sec\theta$

$\theta = \sec^{-1} 2u$

$= \sec^{-1}(2\sin x) + c$

$\Rightarrow$ The most proper trigonometric substitutable to solve.

$\int \frac{dx}{\sqrt{x^2-4x+5}}$ is:-

اكمال م.ت $\longleftarrow$ $\longrightarrow x = \tan\theta + 2$

Lecture 7: Improper integral

#First

$\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$, $\int_{-\infty}^\infty f(x)\,dx$, $\int_a^b f(x)\,dx$, $f(x)$ is cont

حالة ١١ ‎ حالة ١٢

***case 1:** let $f(x)$ cont on $[a,\infty)$.

then, $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$

- exist & equal to M $\Downarrow$ $\int_a^\infty f(x)$ conv to M
- d.n.e $\Downarrow$ $\int_a^\infty f(x)$ diverge

***case 2:** let $f(x)$ cont on $(-\infty, b]$

then, $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$

- exist & equal to M $\Downarrow$ $\int_{-\infty}^b f(x)\,dx$ converge to M
- d.n.e $\Downarrow$ $\int_{-\infty}^b f(x)\,dx$ diverge

***case 3:** let $f(x)$ cont on $(-\infty,\infty)$

then $\int_{-\infty}^\infty f(x)\,dx = \underbrace{\int_{-\infty}^a f(x)\,dx}_{I_1} + \underbrace{\int_a^\infty f(x)\,dx}_{I_2}$

- if $I_1 + I_2$ conv $\longrightarrow \int_{-\infty}^\infty$ conv to $M+L$

  "S.T $I_1 \to M$
  $I_2 \to L$ "

- if I_1 or I_2 div $\Longrightarrow \int_{-\infty}^\infty f(x)$ div lecture7: Improper integral

18/Mar/2022

#First

$$* \int_0^{\infty} e^{-4x} dx = \lim_{t\to\infty} \int_0^t e^{-4x} dx$$

$$= \lim_{t\to\infty} \frac{e^{-4x}}{-4} \Big]_0^t$$

$$= \lim_{t\to\infty} \frac{-1}{4} [e^{-4t} - 1]$$

$$= \frac{-1}{4}[0-1] = \frac{1}{4}$$

$\therefore I = \int_0^{\infty} e^{-4x} . dx$ conv to $\frac{1}{4}$

$$* \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

$$= \underbrace{\int_{-\infty}^{0}}_{I_1} + \underbrace{\int_{0}^{\infty}}_{I_2}$$

$$\cdot I_1 = \int_{-\infty}^{0} \frac{dx}{x^2+1} = \lim_{t\to-\infty} \int_t^0 \frac{dx}{x^2+1} = \lim_{t\to-\infty} \tan^{-1} x \Big]_t^0$$

$$= \lim_{t\to-\infty} [\tan^{-1} 0 - \tan^{-1} t] = 0 - \left(\frac{-\pi}{2}\right) = \frac{\pi}{2}$$

$\therefore I_1$ conv to $\pi/2$

$$\cdot I_2 = \int_0^{\infty} \frac{dx}{x^2+1} = \lim_{t\to\infty} \int_0^t \frac{dx}{x^2+1} = \lim_{t\to\infty} \tan^{-1} x \Big]_0^t$$

$$= \lim_{t\to\infty} [\tan^{-1} t - \tan^{-1} 0] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$\therefore I_2$ conv to $\pi/2$.

$\Rightarrow \therefore I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$ conv to $\frac{\pi}{2} + \frac{\pi}{2} = \pi$

* case 4: If $f(x)$ dis cont at $x=a$ then,

$$\int_a^b f(x).dx = \lim_{t\to a^+} \int_t^b f(x).dx$$

exist = conv   dne = div

* case 5: If $f(x)$ dis cont at $x=b$ then,

$$\int_a^b f(x).dx = \lim_{t\to b^-} \int_a^t f(x).dx$$

exist = conv   dne = div

* case 6: If $f(x)$ dis cont at $x = c \in (a,b)$

then, $\int_a^b f(x).dx = \underbrace{\int_a^c f(x).dx}_{I_1} + \underbrace{\int_c^b f(x).dx}_{I_2}$

. if $I_1 + I_2$ conv $\Rightarrow \int_a^b f(x).dx$ conv.

. if I_1 or I_2 div $\Rightarrow \int_a^b f(x).dx$ div.

lecture 7: Improper Integral

19/Mar/20

First

$$* \int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$$

$$= \underbrace{\int_0^1 \frac{dx}{\sqrt{x}(x+1)}}_{I_1} + \underbrace{\int_1^\infty \frac{dx}{\sqrt{x}(x+1)}}_{I_2}$$

$\bullet \int \frac{dx}{\sqrt{x}(x+1)}$, $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx$

$dx = 2\sqrt{x}\, du$

$\Rightarrow \int \frac{2\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}(u^2+1)}$

$= 2\tan^{-1}\sqrt{x} + c$

$$\bullet I_1 = \int_0^1 \frac{dx}{\sqrt{x}(x+1)} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt{x}(x+1)} = \lim_{t\to 0^+} 2\tan^{-1}\sqrt{x}\Big]_t^1$$

$$= \lim_{t\to 0^+}(2\tan^{-1}1 - 2\tan^{-1}\sqrt{t})$$

$$= (2\cdot\frac{\pi}{4} - 0)$$

$$= \frac{\pi}{2} \ \#$$

$$\bullet I_2 = \int_1^\infty \frac{dx}{\sqrt{x}(x+1)} = \lim_{t\to\infty} \int_1^t \frac{dx}{\sqrt{x}(x+1)} = \lim_{t\to\infty} 2\tan^{-1}\sqrt{x}\Big]_1^t$$

$$= \lim_{t\to\infty} 2\tan^{-1}\sqrt{t} - 2\tan^{-1}\sqrt{1}$$

$$= 2\cdot\frac{\pi}{2} - 2\cdot\frac{\pi}{4}$$

$$I_2 = \frac{\pi}{2}$$

$$\therefore I = \int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

converge to π.

lecture7: Improper Integral 20/Mar/2022

First

* $\int_1^{\infty} \frac{1}{x^2}\,dx$

$= \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \frac{-1}{x}\Big]_1^{b} = \lim_{b\to\infty} \frac{-1}{b} + 1 = 1$

$\therefore$ conv to 1.

* $\int_1^{\infty} \frac{1}{x}\,dx$

$= \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|x|\Big]_1^{b} = \lim_{b\to\infty} \ln|b| - \ln|1| = \infty$

$\therefore$ div

- Note that:

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{converge to } \frac{1}{p-1} & , p>1 \\ \text{diverge} & , 0<p\le 1 \end{cases}$$

* $\int_1^{\infty} \frac{1}{x}\,dx = \text{div}$

* $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \text{div}$

* $\int_1^{\infty} \frac{1}{x^5}\,dx = \text{conv to } \frac{1}{4}$

$\Rightarrow$ *-) find the values of p for which the integral converge. $\int_e^{\infty} \frac{1}{x(\ln x)^{2p}}$

$u = \ln x \Rightarrow du = \frac{1}{x}dx$

if $x=\infty \to u=\infty$, $x=e \to u=1$

$\therefore \int_1^{\infty} \frac{\cancel{x}\,du}{\cancel{x}\,u^{2p}} = \int_1^{\infty} \frac{1}{u^{2p}}\,du$

$\therefore$ conv if $2p>1 \Rightarrow p>\frac{1}{2}$

lecture7 : Improper Integral

#First

20/Mar/2022

*-) find all values of p such that the following integrals are improper.

1-) $\int_{-1}^{2} \frac{1}{x+p}\,dx$, $x+p=0 \Rightarrow x=-p$

$-1 \le x \le 2$

$-1 \le -p \le 2$

$-2 \le p \le 1 \quad \therefore p \in [-2,1]$

2) $\int_{-1}^{3} \frac{dx}{2x-p}$, $2x-p=0 \Rightarrow x=p/2$

$-1 \le x \le 3$

$-1 \le \frac{p}{2} \le 3$

$-2 \le p \le 6 \quad \therefore p \in [-2,6]$

3-) $\int_{-3}^{2} \frac{dx}{x^2-p}$, $x^2-p=0 \rightarrow x^2=p$

$-3 \le x \le 2$

$0 \le x^2 \le 9$

$0 \le p \le 9 \quad \therefore p \in [0,9]$

*find opositive value of p such that $\int_{0}^{\infty} \frac{dx}{x^2+p^2}$ conv to 1

$I = \int_{0}^{\infty} \frac{dx}{x^2+p^2} = \lim_{b \to \infty} \frac{1}{p} \tan^{-1}\left(\frac{x}{p}\right)\Big]_0^b = 1$

$\Rightarrow \frac{1}{p}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right] = 1$ conv to 1

$\Rightarrow \frac{1}{p} \cdot \frac{\pi}{2} - 0 = 1$

$\frac{\pi}{2p} = 1 \Rightarrow p = \pi/2$

lecture7: Improper Integral — 20/Mar/202

First

* $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ conv or div?

$= \lim_{b \to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}} = \lim_{b \to 1^-} \sin^{-1} x \Big]_0^b$

$= \lim_{b \to 1^-} \sin^{-1} b - \sin^{-1} 0$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$ $\therefore$ conv to $\pi/2$

* $\int_{-\infty}^{0} x e^x \, dx$ conv or div

u	dv
x	e^x
1	e^x (+)
0	e^x (−)

$= \lim_{b \to -\infty} \int_b^0 x e^x \, dx$

$= \lim_{b \to -\infty} (x e^x - e^x) \Big]_b^0$

$= \lim_{b \to -\infty} (0 e^0 - e^0 - b e^b + e^b)$

تعبير شكل $0 \cdot \infty$ $(e^{-\infty})$

$= \lim_{b \to -\infty} -1 - b e^b + 0$

$= -1 - \lim_{b \to -\infty} \frac{b}{e^{-b}}$ L.H لوبيتال

$= -1 - \lim_{b \to -\infty} \frac{1}{-e^{-b}}$

$= -1 - 0$

$= -1$ $\therefore$ conv to -1

lecture7: Improper Integral

First

$* \int_0^1 \ln x \, dx$ conv or div

$= \lim_{b \to 0^+} \int_b^1 \ln x \, dx$

$= \lim_{b \to 0^+} x \ln x - x \Big]_b^1$

$= \lim_{b \to 0^+} (1 \ln 1 - 1) - (b \ln b - b)$

نتركه $0 \cdot -\infty$

$= \lim_{b \to 0^+} -1 - \lim_{b \to 0^+} b \ln b - 0$

لوبيتال

$= -1 - \lim_{b \to 0^+} \frac{\ln b}{1/b}$

$= -1 - \lim_{b \to 0^+} \frac{1/b}{-1/b^2}$

$= -1 - \lim_{b \to 0^+} -b$

$= -1 - 0 = -1$ $\therefore$ conv to -1

lecture 8: Improper Integral 21/Mar/2022

#First

• Length of a plane curve, suppose that $y=f(x)$ is a smooth curve on $[a,b]$

"i.e $f'(x)$ cont on $[a,b]$", then the arc length is defined as:

$$L=\int_a^b \sqrt{1+(f'(x))^2}\,.dx$$

* find the arc length of the curve $y=x^{3/2}$ from $(1,1)$ to $(2,2\sqrt{3})$

① $y'=\frac{3}{2}x^{1/2}$

② $(y')^2=\frac{9}{4}x$

③ $L=\int_1^2 \sqrt{1+\frac{9}{4}x}\,.dx$

⋮

* Setup the integral for the arc length of the curve $y=\frac{x^2}{4}-\frac{1}{2}\ln x$ on $[1,2]$ about x-axis:-

① $y'=\frac{2x}{4}-\frac{1}{2}\cdot\frac{1}{x}$

② $(y')^2=(\frac{x}{2}-\frac{1}{2x})^2=\frac{x^2}{4}-2\cdot\frac{1}{2x}\cdot\frac{x}{2}+\frac{1}{4x^2}$

$=\frac{x^2}{4}-\frac{1}{2}+\frac{1}{4x^2}$

③ $(y')^2+1=\frac{x^2}{4}-\frac{1}{2}+\frac{1}{4x^2}+1$

$=\frac{x^2}{4}+\frac{1}{2}+\frac{1}{4x^2}$

$=(\frac{x}{2}+\frac{1}{2x})^2$

④ $L=\int_1^2\sqrt{(\frac{x}{2}+\frac{1}{2x})^2}\,dx=\int_1^2 \frac{x}{2}+\frac{1}{2x}\,.dx$

Five Apple

lecture 8: Improper Integral 22/Mar/2022
#First

• Area of surface of revolution.

let $f(x)$ smoth, non negative on $[a,b]$, then the surfaces area S of the surface of revlution generated by revolving the portion of the curve $y=f(x)$ between $x=a$ and $x=b$ about x-axis :- $S=\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$

* find the area of the of the surface generated by revolving $y=\sqrt{x}$ about x-axis on $[1,4]$

① $y'=\frac{1}{2\sqrt{x}}$ ② $(y')^2=\frac{1}{4x}$

③ $S=\int_1^4 2\pi\cdot\sqrt{x}\cdot\sqrt{1+\frac{1}{4x}}\cdot dx \Rightarrow S=\int_1^4 2\pi\sqrt{x+\frac{1}{4}}\cdot dx$

* find the area of the surface generated by revolving $y=\sqrt{4-x^2}$ on $[-1,1]$ about x-axis :-

① $y'=\frac{-2x}{2\sqrt{4-x^2}}=\frac{-x}{\sqrt{4-x^2}}$

② $(y')^2=\frac{x^2}{4-x^2}$

③ $(y')^2+1=\frac{x^2}{4-x^2}+1$

$=\frac{x^2+4-x^2}{4-x^2}=\frac{4}{4-x^2}$

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10.1: Curves defined by parametric equations.

Defin: the parametric equ of $y=f(x)$ is:

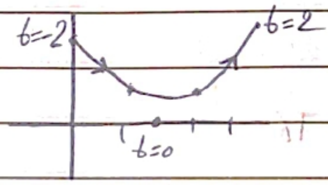
$x=x(t)$, $y=y(t)$, t: independent variable

x, y: dependent variables

Example: Identify the parametric equation

$x=t+2$, $y=t^2$, $-2 \leq t \leq 2$

t	$x(t)$	$y(t)$	(x,y)
-2	0	4	(0,4)
-1	1	1	(1,1)
0	2	0	(2,0)
1	3	1	(3,1)
2	4	4	(4,4)



or: find the cartesian equ

① $t=x-2$

② $y=t^2 \Rightarrow y=(x-2)^2$

if $t=-2 \Rightarrow (x,y)=(0,4)$

if $t=2 \Rightarrow (x,y)=(4,4)$

* Find the cartesian equation for:

Ⓐ1 $x=2t$, $y=t^2+1$, $t \in \mathbb{R}$

$\Downarrow$

$t=\frac{x}{2} \Rightarrow y=(\frac{x}{2})^2+1$

$y=\frac{x^2}{4}+1$

Ⓐ2 $x=t^2+1$, $y=t^4-3$

$\Downarrow$

$t^2=x-1 \Rightarrow y=(x-1)^2-3$

(parabola)

3. $x = 2\sin t$, $y = 2\cos t$, $0 \le t \le 2\pi$

$\frac{x}{2} = \sin t$ $\qquad$ $\frac{y}{2} = \cos t$

$\sin^2 t + \cos^2 t = 1$

$\frac{x^2}{4} + \frac{y^2}{4} = 1 \longrightarrow x^2 + y^2 = 4$ (circle equ)

4. $x = 3\sec t$, $y = 4\tan t$, $0 \le t \le 2\pi$

$\frac{x}{3} = \sec t$ $\qquad$ $\frac{y}{4} = \tan t$

$\sec^2 t - \tan^2 t = 1$

$\frac{x^2}{9} - \frac{y^2}{16} = 1$

5. $x = \cos t + 1$, $y = \sin^2 t$

$\cos t = x - 1$ $\qquad$ $\sin^2 t = 1 - \cos^2 t$

$= 1 - (x-1)^2$ (parabola)

6.) $x = \sec t$, $y = \sin^2 t - 2$

$x = \frac{1}{\cos t} \longrightarrow \cos t = \frac{1}{x}$

- $y = \sin^2 t - 2$

$y = 1 - \cos^2 t - 2$

$= -1 - \cos^2 t$

$= -1 - \frac{1}{x^2}$

7.) $x = 2\sin^2 t$, $y = 3\cos^2 t$, $0 \le t \le 2\pi$

$\frac{x}{2} = \sin^2 t$ $\qquad$ $\frac{y}{3} = \cos^2 t$

$\sin^2 t + \cos^2 t = 1$

$\frac{x}{2} + \frac{y}{3} = 1$

lecture 9: parametric equation - part 1

8-) $x = \cosh t$, $y = \sinh t$

$\cosh^2 t - \sinh^2 t = 1$

$x^2 - y^2 = 1$

9-) $x = t^2 - 2t$, $y = t + 1$, $t \in R$

$\downarrow$

$t = y - 1$

$\therefore x = t^2 - 2t$

$= (y-1)^2 - 2(y-1)$

$= y^2 - 4y - 3$

10-) $x = e^{-t}$, $y = e^{2t}$, $0 \le t \le \ln 8$

$\ln x = -t \rightarrow t = -\ln x$

$\therefore y = e^{2t} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$

$\therefore y = \frac{1}{x^2}$

if $t = 0 \rightarrow x = e^0 = 1$, $y = e^0 = 1$ $(1,1)$

if $t = \ln 8 \rightarrow x = e^{-\ln 8} = \frac{1}{8}$, $y = e^{2\ln 8} = 64$ $(\frac{1}{8}, 64)$

$\therefore y = \frac{1}{x^2}$ from $(1,1)$ to $(\frac{1}{8}, 64)$

* Sketch the curve :-

a-) $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$

$\cos t = \frac{x}{2}$ $\qquad$ $\sin t = \frac{y}{2}$

$\sin^2 t + \cos^2 t = 1$

$\frac{y^2}{4} + \frac{x^2}{4} = 1 \rightarrow x^2 + y^2 = 4$ circle centered by with $r = 2$

counter clockwise

$t = 0 \rightarrow (x,y) = (2,0)$ ← "حتى نعرف

$t = \frac{\pi}{2} \rightarrow (x,y) = (0,2)$ إذا مع عقارب الساعة

← أو عكس"

lecture 9: parametric equation - part 1

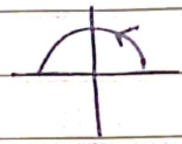
b-) $x = 2\cos t \ , \ y = 2\sin t \ , \ 0 \le t \le \pi$

semi circle

(الأطراف وبنرسم) $t = 0 \rightarrow (x,y) = (2,0)$

$t = \frac{\pi}{2} \rightarrow (x,y) = (0,2)$

$t = \pi \rightarrow (x,y) = (-2,0)$

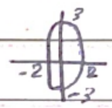


مش مطلوب ابدا

*** Sketch the curve:**

$x = 2\cos t \ , \ y = 3\sin t \ , \ 0 \le t \le 2\pi$

$\therefore \frac{x^2}{4} + \frac{y^2}{9} = 1$



(ellipse)

*** The parametric equ for $9x^2 + 4y^2 = 36$**

$\therefore \frac{9x^2}{36} + \frac{4y^2}{36} = \frac{36}{36}$

$\therefore \frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow x = 2\cos t \qquad 0 \le t \le 2\pi$

$y = 3\sin t$

$\sqrt{4}$, $\sqrt{9}$

• The parametric equation for lines:

1-) x-axis: $x(t) = t \ , \ y(t) = 0 \ , \ t \in \mathbb{R}$

2-) y-axis: $x(t) = 0 \ , \ y(t) = t \ , \ t \in \mathbb{R}$

3-) The line $x = a$: $x(t) = a \ , \ y(t) = t \ , \ t \in \mathbb{R}$

4-) The line $y = b$: $x(t) = t \ , \ y(t) = b \ , \ t \in \mathbb{R}$

5-) The line $y = ax + b$: $x(t) = t \ , \ y(t) = at + b \ , \ t \in \mathbb{R}$

6-) The line from (x_0, y_0) to (x_1, y_1) is: $x = x_0 + (x_1 - x_0)t$

$y = y_0 + (y_1 - y_0)t$

$0 \le t \le 1$

lecture 9: parametric equation - part 1

*Find the parametric equ for the line from $(-5,-3)$ to $(0,2)$:

$$x(t) = 0 + (-5-0)t = -5t$$
$$y(t) = 2 + (-3-2)t = 2-5t$$
$$0 \leq t \leq 1$$

*The parametric equ for the line $y = 3x - 2$, $0 \leq x \leq 1$ is

$$x(t) = t$$
$$y(t) = 3t - 2$$
$$0 \leq t \leq 1$$

*The parametric equation for semi circle is:

$$x(t) = a + r\cos t$$
$$y(t) = b + r\sin t$$
$$0 \leq t \leq \pi$$

*Find the parametric equation for:

$$(x+1)^2 + (y-2)^2 = 25$$

$$x(t) = -1 + 5\cos t$$
$$y(t) = 2 + 5\sin t$$
$$0 \leq t \leq 2\pi$$

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lecture 10: Calculus with parametric eq (curves) 1/4/2022
section 2.1

*Remark:

Δ for the parametric curve $x(t)=x$, $y(t)=y$. The slope is given by,

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{y'(t)}{x'(t)}$$

Example: find the slope of the tangent line for $x(t)=2\cos t+\sin 2t$; $y(t)=2\sin t+\cos 2t$ at $t=0$

Sol: $\frac{dy}{dx}=\frac{y'(t)}{x'(t)}=\frac{2\cos t-2\sin 2t}{-2\sin t+2\cos 2t}\Big]_{t=0}=\frac{2(1)-0}{0+2}=1$

Example: find the slope for $x(t)=t-t^4$, $y(t)=t^2-t^3$ at $(0,0)$

Δ1 $(0,0)\Rightarrow x(t)=0 \rightarrow t-t^4=0$

$t(1-t^3)=0$

$\Rightarrow t=0, t=1$

Δ2 $y(t)=0 \rightarrow t^2-t^3=0$

$t^2(1-t)=0$

$t=0, t=1$

$\therefore t=0, t=1$

Δ3 slope $=\frac{dy}{dx}=\frac{2t-3t^2}{1-4t^3}$

$\text{slope}\Big]_{t=0}=0$, $\rightsquigarrow \text{slope}\Big]_{t=1}=\frac{2-3}{1-4}=\frac{-1}{-3}=\frac{1}{3}$

Example: find the equation of the tangent line for $x(t)=e^t$, $y(t)=t+e^{-t}$ at $t=0$

1-) $x_0=e^0=1$

2-) $y_0=0+e^0=1$

3-) $\text{slope}\Big|_{t=0} \rightarrow =\frac{1-e^{-t}}{e^t}=\frac{1-e^0}{e^0}=0$

$\therefore$ equ of the tangent line

$y-y_0=m(x-x_0)$

$y-1=0(x-1)$

$y=1$ #

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Lecture 10: 1/4/2022
section 2:

① Recall that the slope of $x = x(t)$, $y = y(t)$ is $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}\Big|_{t=t_0}$

then: a-) the curve has horizontal tangent line if $y'(t) = 0$, $x'(t) \neq 0$

b-) " " " vertical " " " $x'(t) = 0$, $y'(t) \neq 0$

Example: find the value of t such that curve parametric by $x(t) = t^2 - 1$, $y(t) = t^4 - 4t^2$

has: a-) horizontal tangent line.

b-) vertical tangent line.

Sol: 1-) $x'(t) = 2t$, $y'(t) = 4t^3 - 8t$

2-) $x'(t) = 0 \Rightarrow 2t = 0 \Rightarrow t = 0$

$y'(t) = 0 \Rightarrow 4t^3 - 8t = 0$

$4t(t^2 - 2) = 0$

$t = 0$ $\quad$ $t = \pm\sqrt{2}$

3-) Horizontal tangent

if $y'(t) = 0$ but $x'(t) \neq 0$

~~1-) $t = 0 \Rightarrow x'(0) = 0$~~ ✗ zero

2-) $t = \sqrt{2} \Rightarrow x'(\sqrt{2}) = 2\sqrt{2}$ ✓

3-) $t = -\sqrt{2} \Rightarrow x'(-\sqrt{2}) = -2\sqrt{2}$ ✓

∴ the curve has horizontal tangent line

at $t = 2\sqrt{2}$, $t = -2\sqrt{2}$

4-) vertical tangent

if $x'(t) = 0$ but $y'(t) \neq 0$

~~at $t = 0 \Rightarrow y'(0) = 4(0) - 8(0) = 0$~~

∴ there is no vertical tangent.

Lecture 10:

section 2:

1/4/2022

Example: find the value of $t \in [0, \pi]$ such that the curve parametrize by $x = \cos(2t)$, $y = \sin(4t)$ has:

a-) horizontal tangent

b-) vertical tangent

Sol:

1-) $x'(t) = -2\sin 2t$ 2-) $y'(t) = 4\cos 4t$

a-) H.T

$y'(t) = 0 \rightarrow 4\cos 4t = 0$

$4t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$

$t = \underbrace{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}}_{\in [0,\pi]}, \frac{9\pi}{8} \notin [0, \pi]$

Now $x'(\frac{\pi}{8}) \neq 0 \neq -2\sin(\frac{\pi}{4})$

$x'(\frac{3\pi}{8}) \neq 0$, $x'(\frac{5\pi}{8}) \neq 0$, $x'(\frac{7\pi}{8}) \neq 0$

$\therefore t = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

b-) V.T

$x'(t) = 0 \rightarrow -2\sin 2t = 0$

$2t = 0, \pi, 2\pi$

$t = 0, \frac{\pi}{2}, \pi \in [0, \pi]$

$y'(t) \neq 0$ $\therefore y'(0) = 4\cos 4(0) \neq$

$y'(\frac{\pi}{2}) = 4\cos 4\frac{\pi}{2} \neq 0$

$y'(\pi) = 4\cos 4\pi \neq 0$

$\therefore t = 0, \frac{\pi}{2}, \pi$

lecture 10:
section2:

1/4/2022

Example: the parametric curve $x(t) = t^3 - 3t$, $y(t) = t^2 + 6t$ has horizontal tangent at the point:

1-) $x'(t) = 3t^2 - 3$, $y'(t) = 2t + 6$

2-) H.T : $y'(t) = 0 \Rightarrow 2t + 6 = 0$

$t = -3$

$x'(-3) = 3(-3)^2 - 3 = 24 \neq 0$

$\therefore$ horizontal tangent at $t = -3$

point $(x(-3), y(-3)) = (-18, -9)$

3-) let $c: x = x(t)$, $y = y(t)$, $t \in [a, b]$

then the arc length of c is: ?

1-) $x'(t) = 1$ 2-) $y'(t) = y'$

3-) $L = \int_A^B \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

Example: find the arc length of the curve

$c: x(t) = \cos t$, $y(t) = \sin t$, $0 \le t \le 2\pi$

1-) $x'(t) = -\sin t$ 2-) $y'(t) = \cos t$

3-) $L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{2\pi} \sqrt{1}\, dt$

$= 2\pi$

lecture 10:
section 2:

1/4/2022

Example: find the arc length of the curve $c: x(t) = e^t - t,\ y(t) = 4e^{1/2t},\ t \in [0,1]$

Sol: 1-) $x'(t) = e^t - 1$ 2-) $y'(t) = \frac{4}{2} e^{1/2t} = 2e^{1/2t}$ 3-) $(x')^2 + (y')^2 = (e^t - 1)^2 + (2e^{1/2t})^2$

$$= e^{2t} - 2e^t + 1 + 4e^t$$
$$= e^{2t} + 2e^t + 1$$
$$= (e^t + 1)^2$$

4-) $L = \int_0^1 \sqrt{(e^t+1)^2}\, dt$

$$= \int_0^1 |e^t + 1|\, dt$$
$$= e^t + t \Big]_0^1$$
$$= (e^1 + 1) - (e^0 + 0)$$
$$= e$$

4-) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

note that: if $\frac{d^2y}{dx^2} > 0 \Rightarrow$ concave up

$\frac{d^2y}{dx^2} < 0 \Rightarrow$ concave down

Example: Consider the curve $c: x(t) = t^2,\ y = t^3 - 3t$ determine whether the curve concave up or down.

1-) $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2 - 3}{2t}$

2-) $\frac{d^2y}{dx^2} = \frac{\frac{(2t)(6t) - (3t^2-3)(2)}{4t^2}}{2t}$

$$= \frac{12t^2 - 6t^2 + 6}{8t^3} = \frac{6t^2 + 6 \to +0}{8t^3 \to t=0}$$

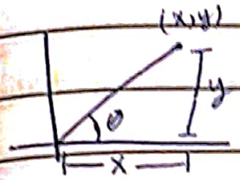
$- - -\ \underset{0}{|}\ + + +$

∴ ~~concave up $(0, \infty)$~~

∴ ~~concave down $(-\infty, 0)$~~

lecture 10: polar coordinates
section 3:

2/4/2022



(x,y): rectangular coordinate
"cartesian"
(r,θ): polar coordinate

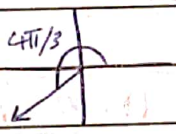
① $(r,\theta) \to (x,y)$

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

Example: Given that the polar coordinate $(6, \frac{4\pi}{3})$ find the rectangular coordinate?

1-) $r = 6, \ \theta = \frac{4\pi}{3}$

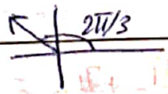


2-) $x = 6\cos(\frac{4\pi}{3}) = 6(\frac{-1}{2}) = -3$

3-) $y = 6\sin(\frac{4\pi}{3}) = 6(\frac{-\sqrt{3}}{2}) = -3\sqrt{3}$

Example: find the cartesian coordinate (x,y) of the point $(2, \frac{2\pi}{3})$?

1-) $r = 2, \ \theta = \frac{2\pi}{3}$



2-) $x = 2\cos(\frac{2\pi}{3}) = -1$

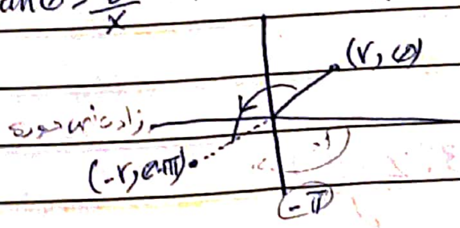
3-) $y = 2\sin(\frac{2\pi}{3}) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$

$\therefore (x,y) = (-1, \sqrt{3})$

* $(x,y) \to (r,\theta)$

$r^2 = x^2 + y^2 \Rightarrow r = \pm\sqrt{x^2+y^2}$

$\tan\theta = \frac{y}{x} \to \theta = \tan^{-1}(\frac{y}{x})$



all polar coordinates:

if $r > 0$:

$(r,\theta) = (r, \theta + 2n\pi)$

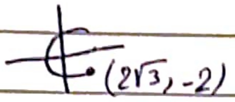
$n \in \mathbb{Z}$

if $r < 0$:

$(-r, \theta+\pi) = (-r, \theta + 2n\pi)$

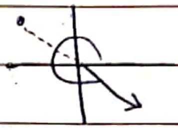
lecture: polar coordinates 2/4/2022
Section 8.1

Example: find all polar coordinates for $(2\sqrt{3}, -2)$

1.) 

2.) $r = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{12+4} = 4$

3.) $\theta = \tan^{-1}\left(\frac{-2}{2\sqrt{3}}\right) = -\tan\left(\frac{1}{\sqrt{3}}\right) = -\pi/6$

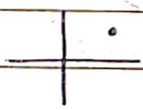
$\therefore (r, \theta) = (4, -\pi/6)$ 

all polar:

4.) if $r>0$: $\left(4, \frac{-\pi}{6} + 2n\pi\right), n \in \mathbb{Z}$

if $r<0$: $\left(-4, \frac{-\pi}{6} + \pi + 2n\pi\right) = \left(-4, \frac{5\pi}{6} + 2n\pi\right), n \in \mathbb{Z}$

Example: find the polar coordinates (r, θ) of the point $(4, 4\sqrt{3})$, where $r<0$ and $0 \le \theta \le 2\pi$

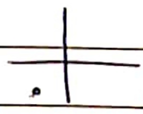


1.) $r = \sqrt{(4)^2 + (4\sqrt{3})^2} = 8$

2.) $\theta = \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right) = \frac{\pi}{3}$ $\therefore (8, \pi/3)$ but $r>0$

3.) $r<0$, $0 \le \theta \le 2\pi$ $\therefore \left(-8, \frac{\pi}{3} + \pi\right) = (-8, 4\pi/3)$

Example: The polar coordinates (r, θ) of the cartesian point $(-3, -3)$, $r<0$, $\theta \in [0, 2\pi]$ is given by:



1.) $r = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$

2.) $\theta = \tan^{-1}\left(\frac{-3}{-3}\right) = \tan^{-1}(1) = 5\pi/4$
في الربع الثالث

$\therefore (3\sqrt{2}, 5\pi/4)$ $r>0$

but $r<0$ $\therefore \left(-3\sqrt{2}, \frac{5\pi}{4} - \pi\right) = \left(-3\sqrt{2}, \frac{\pi}{4}\right)$

bec $\theta \in [0, 2\pi]$

lecture 11: polar coordinates
section 3!

2/4

Example: Express the following equation in polar coordinates.

1-) $x^2+y^2=4$

$r^2=4 \Rightarrow r=\pm 2$

2-) $x^2+y^2+7y=0$

$r^2+7r\sin\theta=0$

3-) $4xy=8$

$4r\cos . r\sin\theta=8$

$2.2.r^2\cos\theta\sin\theta=8$

$2r^2\sin 2\theta=8$

$r^2=\frac{4}{\sin 2\theta} \Rightarrow r^2=4\csc 2\theta$

4-) $(x^2+y^2)^2=x^2-y^2$

$(r^2)(r^2)=r^2\cos^2\theta-r^2\sin^2\theta$

$\cancel{r^2}.r^2=\cancel{r^2}[\cos^2\theta-\sin^2\theta]$

$r^2=\cos 2\theta$, find domain θ??

$r=\pm\sqrt{\cos 2\theta}$ $\therefore$ $\cos 2\theta\geq 0$

$-\frac{\pi}{2}\leq 2\theta\leq\frac{\pi}{2}$

$-\frac{\pi}{4}\leq\theta\leq\frac{\pi}{4}$

5-) $y=\frac{1}{\sqrt{3}}x$

$\frac{y}{x}=\frac{1}{\sqrt{3}}$

$\tan\theta=\frac{1}{\sqrt{3}}$

lecture 11: polar coordinates   2/4/2022
sections:

Example: find the cartesian equation for:

1-) $r = \cos\theta$

$r^2 = r\cos\theta$

$x^2 + y^2 = x$

2-) $r^2 = \sin 2\theta$

$r^2 = 2\sin\theta\cos\theta$

$r^2 \cdot r \cdot r = 2 \cdot r\sin\theta \cdot r\cos\theta$

$(r^2)^2 = 2yx$

$(x^2+y^2)^2 = 2yx$

3-) $\frac{r}{1} = \frac{6}{3-\sin\theta}$

$3r - r\sin\theta = 6$

$3r = 6 + r\sin\theta$

$9r^2 = (6 + r\sin\theta)^2$

$9(x^2+y^2) = (6+y)^2$

4-) $r = \sec\theta\tan\theta$

$r = \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta}$

$r = \frac{\sin\theta}{\cos^2\theta}$

$r\cos^2\theta = \sin\theta$

$r\cos x$
$r\cos x$ $\quad r^2\cos^2\theta = r\sin\theta$

$x^2 = y$

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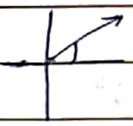
lecture 12: Area in polar coordinates 14/4/2022

section 4:

1-) Radical line

$\theta = \theta_0$, eg: $\theta = \frac{\pi}{4}$



2-) General line

$r = \frac{c}{a\cos\theta + b\sin\theta}$, a, b, c constants

$ar\cos\theta + br\sin\theta = c$

$ax + by = c$

Example: find the slope of $r = \frac{3}{2\cos\theta + 4\sin\theta}$

1-) $2r\cos\theta + 4r\sin\theta = 3$

$2x + 4y = 3$

$4y = 3 - 2x$

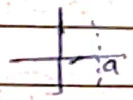
$y = \frac{3}{4} - \frac{x}{2}$

2-) $\underset{(x \text{ dalso})}{\text{slope}} = \frac{-1}{2}$

Note that:

$a = r\cos\theta$ ($x = a$)

1-) $r = \frac{a}{\cos\theta} \Rightarrow a\sec\theta$ "vertical line $x = a$"



2-) $r = \frac{b}{\sin\theta} = b\csc\theta$ "Horizontal line $y = b$"

$r\sin\theta = b$

$y = b$

lecture 18: Area in polar coordinates 14/4/2022
section 4.3

(3) Circles

$r = 2a\cos\theta + 2b\sin\theta$

circle: center (a,b), radius $r = \sqrt{a^2+b^2}$

Show that:

$r = 2a\cos\theta + 2b\sin\theta$

$r^2 = 2ar\cos\theta + 2br\sin\theta$

$x^2 + y^2 = 2ax + 2bx$

$(x^2 - 2ax) + (y^2 - 2by) = 0$

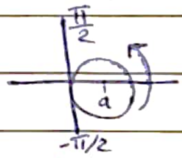
$(x^2 - 2ax + a^2) + (y^2 - 2by + b^2) = a^2 + b^2$

$(x-a)^2 + (y-b)^2 = a^2 + b^2$

Example: the center of the circle $r = 4\sin\theta - 3\cos\theta$ is:

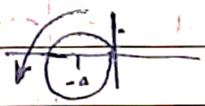
center $\left(\frac{-3}{2}, \frac{4}{2}\right)$
x , y

Note that: 1-) $r = 2a\cos\theta$ circle, center $(a,0)$ $r = a$



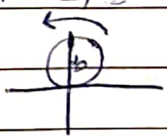
$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$
counter clockwise.

2-) $r = -2a\cos\theta$ circle, center $(-a, 0)$, $r = a$



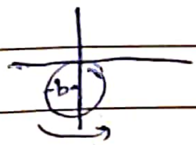
$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

3-) $r = 2b\sin\theta$ circle, center $(0,b)$, $r = b$



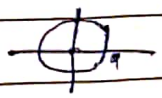
$0 \le \theta \le \pi$

4-) $r = -2b\sin\theta$ circle, center $(0,-b)$, $r = b$



$\pi \le \theta \le 2\pi$

5-) $r = a$



circle, center $(0,0)$, $r = a$
$0 \le \theta \le 2\pi$

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lecture 12: Area in polar coordinates
section 4.1

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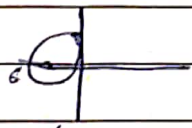
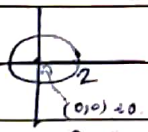
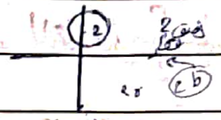
Example:



$r = -6\cos\theta$



$r = 2$

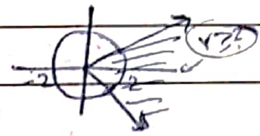


$r = 4\sin\theta$

Example: Sketch the region $2 < r \leq 3$

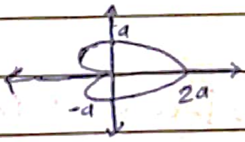
Example: Sketch the region $r \geq 2$, $|\theta| \leq \frac{\pi}{4}$

$\frac{-\pi}{4} \leq \theta \leq \frac{\pi}{4}$

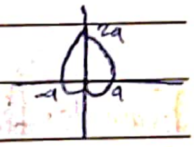


$\boxed{4}$ Cardioid

1-) $r = a(1+\cos\theta)$
$= -a(1-\cos\theta)$



3-) $r = a(1+\sin\theta)$
$r = -a(1-\sin\theta)$



How? $(r,\theta) = (-r, \theta+\pi)$

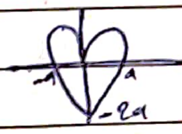
$r = a(1+\cos\theta)$

$-r = a(1+\cos(\theta+\pi))$

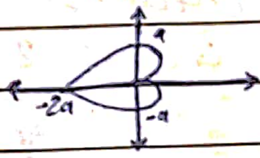
$r = -a(1+[\cos\theta\cos\pi - \sin\theta\sin\pi])$

$r = -a(1-\cos\theta)$

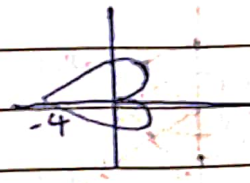
4-) $r = a(1-\sin\theta)$
$= -a(1+\sin\theta)$



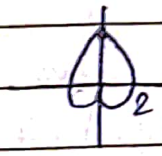
2) $r = a(1-\cos\theta)$
$= -a(1+\cos\theta)$



Example:



$r = -2(1+\cos\theta)$
or
$r = 2(1-\cos\theta)$



$r = -2(1-\sin\theta)$
or
$r = 2(1+\sin\theta)$

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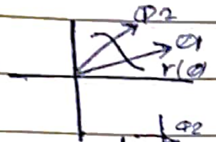
lecture 12: Area in polar curves — 22/4/2022

section 4.1

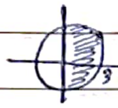


$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} (r(\theta))^2 \, d\theta$$

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} (\overset{\text{الأبعد}}{r_2(\theta)})^2 - (\overset{\text{الأقرب}}{r_1(\theta)})^2 \, d\theta$$

Example: find the area of the region in the right half plane and inside the circle $r=3$

sol:



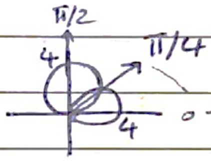
$$\rightarrow A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (3)^2 \, d\theta = 9(\pi/2 - -\pi/2) = 9\pi$$

Example: find the area of the region that is common to the circles $r = 4\cos\theta$, $r = 4\sin\theta$

Sol: $4\sin\theta = 4\cos\theta$

$1 = \tan\theta$

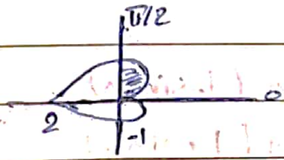
$\frac{\pi}{4} = \theta$



$$\rightarrow A = \frac{1}{2}\int_0^{\pi/4} (4\sin\theta)^2 + \int_{\pi/4}^{\pi/2} (4\cos\theta)^2 \, d\theta$$

Example: find the area of the region inside $r = 1-\cos\theta$ of in the first quadrant?

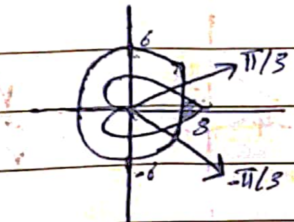
sol:



$$\rightarrow A = \frac{1}{2}\int_0^{\pi/2} (1-\cos\theta)^2 \, d\theta = \frac{3\pi}{8} - 1$$

Example: find the area of the region that is inside $r = 4+4\cos\theta$ and outside $r=6$

sol:



$4+4\cos\theta = 6$

$4\cos\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \pm\pi/3$

$$A = \underline{\underline{2}} * \frac{1}{2}\int_0^{\pi/3} (4+4\cos\theta)^2 - (6)^2 \, d\theta = 18\sqrt{3} - 4\pi$$

lecture 12: Area in polar

section 4:

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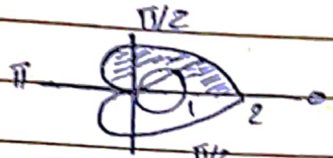
Example: find the area of the region in the upper half plane outside the circle $r = \cos\theta$ and inside the cardi $r = 1+\cos\theta$

Sol:



$1+\cos\theta = \cos\theta$

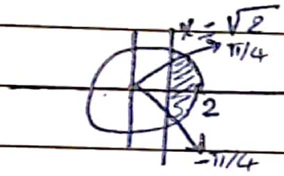
$1 \neq 0$

$$A = \frac{1}{2}\int_0^{\pi/2}\left[(1+\cos\theta)^2 - (\cos\theta)^2\right]d\theta + \frac{1}{2}\int_{\pi/2}^{\pi}(1+\cos\theta)^2\,d\theta$$

Example: find the area of the region that is inside $r = 2$ and to right of $r\cos\theta = \sqrt{2}$

sol: $r = 2$, $r = \frac{\sqrt{2}}{\cos\theta}$

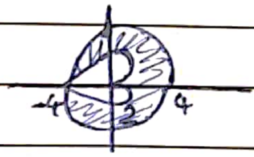
$$2 = \frac{\sqrt{2}}{\cos\theta} \Rightarrow \cos\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \pm\pi/4$$



$$\rightarrow A = 2 * \frac{1}{2}\int_0^{\pi/4}\left[(2)^2 - \left(\frac{\sqrt{2}}{\cos\theta}\right)^2\right]d\theta = \pi - 2$$

Example: find the area of the region outside $r = 2-2\cos\theta$ and inside $r = 4$

sol: $r = 2(1-\cos\theta)$, $r = 4$



$4 = 2(1-\cos\theta)$

$2 = 1-\cos\theta$

$\cos\theta = -1$

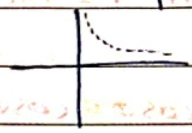
$\theta = \pi$

$$A = 2 * \frac{1}{2}\int_0^{\pi}\left[(4)^2 - (2-2\cos\theta)^2\right]d\theta = 10\pi$$

Lecture 13: Sequences part1 $\boxed{11.1}$ 23/4/2022

(e.g) sketch $f(x) = \frac{1}{x}$, $x \in \mathbb{N}$

(e.g) sketch $f(x) = \frac{1}{x}$, $x \in \mathbb{N}$



Remark: A sequence of real number is just a function whose domain is "$\mathbb{N}$" denoted by $\{f(n)\}_{n=1}^{\infty}$

$f(n) : \mathbb{N} \to \mathbb{R}$

$(a_n)_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$, $a_n, n \geq 1$

Example: given that $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\}$

find the general form:

$a_1 = \frac{1}{2}$

$a_2 = \frac{2}{3}$

$a_3 = \frac{3}{4}$

$\therefore a_n = \{\frac{n}{n+1}\}_{n=1}^{\infty}$

Example: $\{\frac{1}{n}\}_{n=1}^{\infty}$ then $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $a_4 = \frac{1}{4}$

$= \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

Example: $\{\underset{a_1}{1}, \underset{a_2}{2}, \underset{a_3}{4}, \dots\} = \{2^{n-1}\}_{n=1}^{\infty}$ ↳ General form.

Examples: $\{-1, 1, -1, 1, -1, 1, \dots\} = \{(-1)^n\}_{n=1}^{\infty}$

Theorem: the sequence $\{a_n\}_{n=1}^{\infty}$ converge to L if and only if $\lim\limits_{n\to\infty} a_n = L$

iff $\iff$

If $\lim\limits_{n\to\infty} a_n$ d.n.e $\Rightarrow \{a_n\}$ diverge.

lecture 13: Sequences part 1

$\boxed{11.1}$

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Example: Determine whether the following seq. conv or div?

* اذا طلب conv
يعني بدو نهاية والعكس

1-) $\{3^n\}_{n=1}^{\infty}$, $\lim\limits_{n\to\infty} (3^n) = \infty$ $\therefore$ div

2-) $\{\cos(\frac{\pi}{n})\}_{n=1}^{\infty}$, $\lim\limits_{n\to\infty} \cos(\frac{\pi}{n}) = \cos(0) = 1$ $\therefore$ conv to 1

3-) $\left\{\frac{3n^2+5}{2n^2-1}\right\}_{n=1}^{\infty}$, $\lim\limits_{n\to\infty} \frac{3n^2+5}{2n^2-1} = \frac{3}{2}$ $\therefore$ conv to $\frac{3}{2}$

4-) $\left\{\frac{(1+n)^3}{3n^2+2}\right\}_{n=1}^{\infty}$, $\lim\limits_{n\to\infty} \frac{(1+n)^3}{3n^2+2} = \infty$ $\therefore$ div

5-) $\{\ln n\}_{n=1}^{\infty}$, $\lim\limits_{n\to\infty} \ln n = \infty$ $\therefore$ div

6-) $\left\{\frac{n^2+10n+3}{e^{2n}}\right\}_{n=1}^{\infty}$

$= \lim\limits_{n\to\infty} \frac{n^2+10n-5}{e^{2n}}$ $\quad \frac{\infty}{\infty}$ لوبيتال

L'H $\lim\limits_{n\to\infty} \frac{2n+10}{2e^{2n}} = \frac{\infty}{\infty}$

L'H $\lim\limits_{n\to\infty} \frac{2}{4e^{2n}} = 0$ $\therefore$ conv to zero

Note that:

1-) $\lim\limits_{n\to\infty} \left(1+\frac{a}{n}\right)^{bn} = e^{ab}$, $a \in \mathbb{R}$

2-) $\lim\limits_{x\to\infty} \frac{e^x}{x^n} = \infty$

3-) $\lim\limits_{x\to\infty} \frac{x^n}{e^x} = 0$

7-) $\left\{\left(1+\frac{1}{n}\right)^{-3n}\right\}_{n=1}^{\infty}$, $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^{-3n} = \lim\limits_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^{-3} = e^{-3}$ $\therefore$ conv to e^{-3}

Lecture 13: Sequnces part 1 23/4/2022

$\boxed{11.1}$

8.) $\{n\sin(\frac{\pi}{n})\}_{n=1}^{\infty}$, $\lim_{n\to\infty} n\sin(\frac{\pi}{n}) = \infty\cdot 0$ (مالم)

$= \lim_{n\to\infty} \frac{\sin(\pi/n)}{1/n} = \frac{0}{0}$

$\Rightarrow$ L'H $\lim_{n\to\infty} \frac{\cos(\pi/n) * \frac{-\pi}{n^2}}{(\frac{-1}{n^2})}$

$= \lim_{n\to\infty} \pi\cos(\pi/n) = \pi * 1 = \pi$ $\therefore$ conv to π

9.) The limit of the seq $a_n = \ln(n^2+3n) - 2\ln(3n+1)$ is:

$= \lim_{n\to\infty} \ln(n^2+3n) - \ln(3n+1)^2$

$= \lim_{n\to\infty} \ln\left(\frac{n^2+3n}{(3n+1)^2}\right)$ درجة البسط=درجة المقام

$= \ln\left(\lim_{n\to\infty} \frac{n^2+3n}{(3n+1)^2}\right)$

$= \ln\frac{1}{9} = -\ln 9$

10.) limit of the seq $a_n = \sqrt{n^2+n+1} - n$ is:-

$\lim_{n\to\infty} \sqrt{n^2+n+1} - n = \infty - \infty$ (مالم)

$= \lim_{n\to\infty} (\sqrt{n^2+n+1} - n) \frac{(\sqrt{n^2+n+1} + n)}{(\sqrt{n^2+n+1} + n)}$

$= \lim_{n\to\infty} \frac{(n^2+n+1) - n^2}{\sqrt{n^2+n+1} + n}$

$= \lim_{n\to\infty} \frac{n+1}{\sqrt{n^2+n+1} + n}$

$= \lim_{n\to\infty} \frac{n(1+\frac{1}{n})}{|n|\sqrt{1+\frac{1}{n}+\frac{1}{n^2}} + n}$ $\Longrightarrow$

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$$= \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{1}{n}\right)}{\cancel{n}\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+1\right)}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+1} = \frac{1+0}{1+1} = \frac{1}{2} \checkmark \text{ and conv to } \frac{1}{2} \checkmark$$

11-) $\left\{\left(\frac{n+1}{n-7}\right)^{3n}\right\}_{n=1}^{\infty}$ conv or div?

$$\lim_{n\to\infty} \left(\frac{n+1}{n-7}\right)^{3n} = \lim_{n\to\infty} \left(\frac{\frac{n+1}{n}}{\frac{n-7}{n}}\right)^{3n}$$

$$= \lim_{n\to\infty} \left(\frac{\left(1+\frac{1}{n}\right)^{3n}}{\left(1-\frac{7}{n}\right)^{3n}}\right) = \frac{(e^1)^3}{(e^{-7})^3} = \frac{e^3}{e^{-21}} = e^3 \cdot e^{21} = e^{24}$$

$\therefore$ conv to e^{24}

12-) the limit of the seq $a_n = \sqrt[n]{n+1}$ is:

$$= \lim_{n\to\infty} \sqrt[n]{n+1} = \lim_{n\to\infty} (n+1)^{1/n} \quad \infty^0 \text{ (IND)}$$

$$y = (n+1)^{\frac{1}{n}}$$

$$\ln y = \frac{1}{n}\ln(n+1)$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln(n+1)}{n} \quad \frac{\infty}{\infty} \text{ IND}$$

$$L'H \Rightarrow \lim_{n\to\infty} \frac{\frac{1}{n+1}}{1} = 0$$

$e^0 = 1 \Rightarrow$ conv to 1

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$\boxed{11.1}$

13-) the limit of the seq $\{(4n+e^n)^{2/n}\}$ is:-

$$\lim_{n\to\infty}(4n+e^n)^{2/n} \quad \infty^0 \text{ (ratio)}$$

$$y=(4n+e^n)^{2/n}$$

$$\ln y=\frac{2}{n}\ln(4n+e^n)$$

$$\lim_{n\to\infty}\ln y=\lim_{n\to\infty}\frac{2\ln(4n+e^n)}{n} \quad \frac{\infty}{\infty} \text{ (ratio)}$$

$$L'H \lim_{n\to\infty} 2\cdot\frac{4+e^n}{4n+e^n}=\frac{\infty}{\infty}$$

$$L'H \lim_{n\to\infty} 2\cdot\left(\frac{e^n}{4+e^n}\right)=\frac{\infty}{\infty}$$

$$L'H \lim_{n\to\infty} 2\cdot\left(\frac{\cancel{e^n}}{\cancel{e^n}}\right)=2$$

$\Rightarrow e^2$ ∴ conv to e^2

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Note that:

- if $(a_n)_{n=1}^{\infty}$ conv to L then Both (a_{2n}) and (a_{2n+1}) conv to L
  a_{2n}: even terms, a_{2n+1}: odd terms

- if a_{2n+1} "odd terms" conv to L and a_{2n} "even terms" conv to L then $\{a_n\}$ conv to L

- if a_{2n+1} "odd terms" conv to L
  a_{2n} "even terms" conv to M
  then $\{a_n\}$ div

Example: Let $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = \begin{cases} 4+\frac{1}{n}, & n \text{ odd} \\ 2-\frac{1}{n^3}, & n \text{ even} \end{cases}$

is $\{a_n\}$ conv or div?

Sol: 1.) $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} 2-\frac{1}{n^3} = 2$

2.) $\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} 4+\frac{1}{n} = 4$

$\therefore (a_n)$ div

Example: $\{\cos(n\pi)\}_{n=1}^{\infty}$ conv or div?

$a_1=-1,\ a_3=-1,\ a_5=-1$
$a_2=1,\ a_4=1,\ a_6=1$

$= \{-1, 1, -1, 1, -1, 1, \dots\}$

$\therefore (a_{2n}) = 1, \quad a_{2n+1} = -1$

a_{2n} conv to 1, a_{2n+1} conv to -1

$\therefore$ div

lecture (4) sequences part 2 25/4/2022

Example: $\{1, 1, 1, 1, 1, 1, \dots\}$

$\therefore (a_n) = 1$ conv to 1

Example: $\{1 + (-1)^n\}_{n=1}^{\infty}$ conv or div?

$\{0, 2, 0, 2, 0, 2, 0, 2, 0, 2, \dots\}$

a_{2n} conv to 2, a_{2n+1} conv to 0 $\therefore$ div

Example: $\{2, \frac{1}{2}, 4, \frac{1}{4}, 6, \frac{1}{6}, 8, \frac{1}{8}, \dots\}$

a_1 ($1+1$), a_2, a_3 ($1+3$), a_4, a_5 ($1+5$)

$= \begin{cases} n+1, & n \text{ odd} \\ \frac{1}{n}, & n \text{ even} \end{cases}$ $\lim_{n\to\infty} n+1 = \infty$, $\lim_{n\to\infty} \frac{1}{n} = 0$ $\therefore$ div

- All alternating seq $((-1)^n)$ $((-1)^{n+1})$ ...

Theorem: if $\lim_{n\to\infty} = 0 \iff \lim_{n\to\infty} |a_n| = 0$

Example: $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ conv or div?

$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$, $\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \left|\frac{(-1)^n}{n}\right|$ conv to zero

Example: $\left\{\frac{(-1)^{n+1}}{3^n}\right\}_{n=1}^{\infty}$ conv or div?

$\left|\frac{(-1)^{n+1}}{3^n}\right| = \frac{1}{3^n}$, $\lim_{n\to\infty} \frac{1}{3^n} = 0 \Rightarrow$ since $\left|\frac{(-1)^{n+1}}{3^n}\right|$ conv to zero lecture 4: Sequences part 2 26/4/2022

Example: $\left\{\frac{(-1)^{n+1}}{3^n}\right\}_{n=1}^{\infty}$ conv or div?

zero ... even, odd ... a_{2n} a_{2n+1}

$\left|\frac{(-1)^{n+1}}{3^n}\right| = \frac{1}{3^n}$, $\lim_{n\to\infty} \frac{1}{3^n} = 0 \Longrightarrow$ since $\left|\frac{(-1)^{n+1}}{3^n}\right|$ conv to zero

then $\left\{\frac{(-1)^{n+1}}{3^n}\right\}_{n=1}^{\infty}$ conv to zero

Example: $\left\{\frac{(-1)^n(n^2+1)}{2n^2+3n+5}\right\}_{n=1}^{\infty}$ conv or div?

① $|a_n| = \left|\frac{(-1)^n(n^2+1)}{2n^2+3n+5}\right| = \frac{n^2+1}{2n^2+3n+5}$ conv to $\frac{1}{2}$

$\therefore a_{2n} = \frac{(-1)^{2n}((2n)^2+1)}{2(2n)^2+3(2n)+5} = \frac{4n^2+1}{8n^2+6n+5}$ conv to $\frac{1}{2}$

$\therefore a_{2n+1} = \frac{(-1)^{2n+1}((2n+1)^2+1)}{2(2n+1)^2+3(2n+1)+5} = \frac{-4n^2-4n}{8n^2+8n+2+6n+3+5}$ conv to $\frac{-1}{2}$

since odd term conv to $-\frac{1}{2}$
even term conv to $+\frac{1}{2}$ $\therefore$ div

Examples: $\left\{\frac{(-1)^n}{n^2}\right\}_{n=1}^{\infty}$ conv or div?

① $\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$ conv to zero $\therefore \left\{\frac{(-1)^n}{n^2}\right\}$ conv to zero

Examples: $\left\{\frac{(-1)^n n+3}{5n+2}\right\}_{n=1}^{\infty}$ conv or div?

① $|a_n| = \left|\frac{(-1)^n n+3}{5n+2}\right| = \frac{1}{5}$

$\therefore a_{2n} = \frac{(-1)^{2n}(2n)+3}{5(2n)+2} = \frac{2n+3}{10n+2}$ conv to $\frac{1}{5}$

$\therefore a_{2n+1} = \frac{(-1)^{2n+1}(2n+1)+3}{5(2n+1)+2} = \frac{-2n+2}{10n+7}$ conv to $\frac{-1}{5}$

since odd term conv to $-\frac{1}{5}$
even term conv to $\frac{1}{5}$ $\therefore$ div

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Example: $\left\{\frac{(-1)^n}{n^2+n}\right\}_{n=1}^{\infty}$ conv or div?

$\left|\frac{(-1)^n}{n^2+n}\right| = \frac{1}{n^2+n}$ conv to zero

$\therefore \left\{\frac{(-1)^n}{n^2+n}\right\}_{n=1}^{\infty}$ conv to zero

Example: $\left\{\frac{(-1)^n n+n^2}{(3n+1)^2}\right\}$ conv or div

$\left|\frac{(-1)^n n+n^2}{(3n+1)^2}\right|$ conv to $\frac{1}{9}$

$\therefore a_{2n} = \frac{(-1)^{2n}(2n)+(2n)^2}{(3(2n)+1)^2} = \frac{2n+4n^2}{(6n+1)^2}$ conv to $\frac{4}{36} = \frac{1}{9}$

$\therefore a_{2n+1} = \frac{(-1)^{2n+1}(2n+1)+(2n+1)^2}{(3(2n+1)+1)^2} = \frac{-2n-1+4n^2+8n+1}{(6n+3+1)^2}$ conv to $\frac{4}{36} = \frac{1}{9}$

Since even term conv to $\frac{1}{9}$ and odd term to $\frac{1}{9}$

$\therefore \left\{\frac{(-1)^n n+n^2}{(3n+1)^2}\right\}_{n=1}^{\infty}$ conv. to $\frac{1}{9}$

- Sequeezing Thrm

if we ask $(c_n)_{n=1}^{\infty}$ conv or div

and $a_n \le c_n \le b_n$ $\therefore (c_n)_{n=1}^{\infty}$ conv to L

(a_n) conv to L (b_n) conv to L

Example: $\left\{\frac{\sin^2(n)}{n}\right\}_{n=1}^{\infty}$ conv or div

$-1 \le \sin(n) \le 1$

$\lim_{n\to\infty} \frac{\sin^2(n)}{n} = 0$

$0 \le \sin^2(n) \le 1$

$\frac{0}{n} \le \frac{\sin^2 n}{n} \le \frac{1}{n}$

$0 \quad\quad 0 \quad\quad 0$

conv to zero

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Example: $\left\{\frac{n+\sin n}{n}\right\}_{n=1}^{\infty}$ conv or div

$\frac{n+\sin n}{n} = 1 + \frac{\sin n}{n}$ $\qquad -1 \le \sin n \le 1$

$\therefore \left\{\frac{n+\sin n}{n}\right\} = \lim_{n\to\infty} 1 + \frac{\sin n}{n}$ $\qquad \frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$, $\frac{-1}{n} \to 0$, $\frac{1}{n} \to 0$

$1 + 0 = 1$

$\therefore$ conv to 1

Example: $\left\{\frac{3\cos(2n)}{n}\right\}_{n=1}^{\infty}$ conv or div?

$\lim_{n\to\infty} \frac{3\cos(2n)}{n} = 0$ $\qquad -1 \le \cos(2n) \le 1$

$\therefore \left\{\frac{3\cos(2n)}{n}\right\}_{n=1}^{\infty}$ conv to zero $\qquad \frac{-3}{n} \le \frac{3\cos(2n)}{n} \le \frac{3}{n}$, $\frac{-3}{n} \to 0$, $\frac{3}{n} \to 0$

Example: $\{\sin(n)\}_{n=1}^{\infty}$ conv or div

$\lim_{n\to\infty} \sin(n) = \text{d.n.e} \therefore \text{div}$

Example: $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}_{n=1}^{\infty}$ conv or div?

$= \{1, 0, -1, 0, 1, 0, -1, 0, \dots\}$

$\therefore$ even term conv to zero
odd term div $\qquad \therefore$ div

Example: $\{\sin(n\pi)\}_{n=1}^{\infty}$ conv or div

$= \{0, 0, 0, 0, 0, \dots\} \therefore$ conv to zero

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Thrm: $\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$ conv to zero

$$\frac{n}{n}\cdot\frac{(n-1)}{n}\cdot\frac{(n-2)}{n}\cdot\frac{(n-3)}{n}\cdots\frac{3}{n}\cdot\frac{2}{n}\cdot\frac{1}{n}$$

$0 <$ conv to zero

$\leq \frac{1}{n}$ conv to zero

- Geometric seq: $\{r^n\}_{n=1}^{\infty}$

$= \{r, r^2, r^3, r^4, \ldots\}$

$$r^n = \begin{cases} \text{conv if } & -1 < r \leq 1 \\ \text{div if } & r \leq -1,\ r > 1 \end{cases}$$

$r = 1 \to$ conv to 1

$r = -1 \to (-1)^n$ div

$-1 < r <$ conv to zero

Example, $\{(1)^n\}$ conv to 1

Example: $\{(-1)^n\} = \{-1, 1, -1, 1 \ldots\}$ div

Example: $\{(5)^n\}_{n=1}^{\infty}$ div

Example: if $\left\{\frac{a^{n+1}}{2^n}\right\}_{n=1}^{\infty}$ conv find the values of a?

$$\frac{a^{n+1}}{2^n} = a\cdot\frac{a^n}{2^n} = a\left(\frac{a}{2}\right)^n, \text{ conv if } -1 < \frac{a}{2} \leq 1$$

$-2 < a \leq 2$

$\therefore a \in (-2, 2]$

H.W: $\left\{\frac{3^n}{b^n}\right\}_{n=1}^{\infty}$ conv find b?

Example: $\left\{(2^n + 3^n)^{1/2}\right\}_{n=1}^{\infty}$ $\frac{\infty}{\infty}$ L'H

or $\lim_{n\to\infty}\left(3^n\left(\left(\frac{2}{3}\right)^n + 1\right)\right)^{\frac{1}{n}} = \lim_{n\to\infty} 3\cdot\left(\left(\frac{2}{3}\right)^n + 1\right)^{\frac{1}{n}} = 3 * (0+1)^0$

$= 3$

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• Sequence defined recursively

Example: the seq $\{a_n\}_{n=1}^{\infty}$ defined as:

$$a_1 = 2 , \; a_n = a_{n-1} + \frac{1}{a_{n-1}} , \; n \geq 2$$

find the 4-th term?

1.) $a_1 = 2$

2.) $a_2 = a_1 + \frac{1}{a_1} = 2 + \frac{1}{2} = \frac{5}{2}$

3.) $a_3 = a_2 + \frac{1}{a_2} = \frac{5}{2} + \frac{1}{5/2} = \frac{5}{2} + \frac{2}{5} = \frac{29}{10}$

4.) $a_4 = a_3 + \frac{1}{a_3} = \frac{29}{10} + \frac{1}{29/10} = \frac{29}{10} + \frac{10}{29} = \frac{941}{290}$

Example: find the first term of the seq in which $a_n = 2a_{n-1}$ if $a_6 = 64$

1.) $a_6 = 2a_5 \Rightarrow 64 = 2a_5 \Rightarrow \boxed{a_5 = 32}$

2.) $a_5 = 2a_4 \Rightarrow 32 = 2a_4 \Rightarrow \boxed{a_4 = 16}$

3.) $a_4 = 2a_3 \Rightarrow 16 = 2a_3 \Rightarrow \boxed{a_3 = 8}$

4.) $a_3 = 2a_2 \Rightarrow 8 = 2a_2 \Rightarrow \boxed{a_2 = 4}$

5.) $a_2 = 2a_1 \Rightarrow 4 = 2a_1 \Rightarrow \boxed{a_1 = 2}$

Example: ① Note that: Given that the seq $\{a_n\}_{n=1}^{\infty}$ conv to 3 that is $\lim_{n\to\infty} a_n = L = 3$, so

$$\lim_{n\to\infty} a_{n+5} = 3 \quad \text{and} \quad \lim_{n\to\infty} a_{n-4} = 3$$

In General $\lim_{n\to\infty} a_{n+c} = 3$

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Example: Given that $a_1 = \sqrt{6}$, $a_2 = \sqrt{6+\sqrt{6}}$, $a_3 = \sqrt{6+\sqrt{6+\sqrt{6}}}$

a-) find the recursive formula?

1-) $a_1 = \sqrt{6}$

2-) $a_2 = \sqrt{6+\sqrt{6}} = \sqrt{6+a_1}$ $\therefore a_2 = \sqrt{6+a_1}$

3-) $a_3 = \sqrt{6+\sqrt{6+\sqrt{6}}} = \sqrt{6+a_2}$ $\therefore a_3 = \sqrt{6+a_2}$

$\therefore a_{n+1} = \sqrt{6+a_n}$, $a_1 = \sqrt{6}$ $\forall n \geq 1$

b-) given this seq conv find it's limit?

Let $\lim\limits_{n\to\infty} a_n = L \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = L$

Now,, $a_{n+1} = \sqrt{6+a_n}$

$\lim\limits_{n\to\infty} a_{n+1} = \sqrt{6+\lim\limits_{n\to\infty} a_n}$

$L = \sqrt{6+L}$

$L^2 = 6+L \Rightarrow L^2 - L - 6 = 0 \Rightarrow (L-3)(L+2)$

$L=3$ $\quad$ ~~$L=-2$~~

$\therefore$ conv to 3

Example: Given that $a_1 = 1$, $a_n = \frac{1}{2}\left(a_{n-1} + \frac{3}{a_{n-1}}\right)$ $\forall n \geq 2$

find it's limit?

Let $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n-1} = L$

$\therefore \lim\limits_{n\to\infty} a_n = \frac{1}{2}\left(\lim\limits_{n\to\infty} a_{n-1} + \frac{3}{\lim\limits_{n\to\infty} a_{n-1}}\right)$

$L = \frac{1}{2}\left(L + \frac{3}{L}\right)$

$(2L = (L + \frac{3}{L}) * L$

$2L^2 = L^2 + 3$

$L^2 = 3 \Rightarrow L = \pm\sqrt{3} \Rightarrow \therefore L = +\sqrt{3}$ $\therefore \lim\limits_{n\to\infty} a_n = \sqrt{3}$ $\therefore$ conv to $\sqrt{3}$

- Monotone seq :

Monotone : increasing or decreasing

1.) increasing

e.g : $\{n\}_{n=1}^{\infty}$ $\{\underset{a_1}{1}, \underset{a_2}{2}, \underset{a_3}{3}, \underset{a_4}{4}, 5, \dots\}$

$a_1 < a_2 < a_3 < a_4 \dots a_n < a_{n+1}$

(i) $a_{n+1} - a_n > 0$ (ii) $\frac{a_{n+1}}{a_n} > 1$

(iii) $a_n = f(x)$, $\forall x \geq 1$ if $f'(x) > 0 \therefore$ increasing

2.) decreasing

(e.g) : $\{\frac{1}{n}\}_{n=1}^{\infty}$, $\{\underset{a_1}{1}, \underset{a_2}{\frac{1}{2}}, \underset{a_3}{\frac{1}{3}}, \underset{a_4}{\frac{1}{4}}, \underset{a_5}{\frac{1}{5}}, \dots\}$

$a_1 > a_2 > a_3 > a_4 > a_5 > \dots a_n > a_{n+1}$

(i) $a_{n+1} - a_n < 0$ (ii) $\frac{a_{n+1}}{a_n} < 1$ (iii) $a_n = f(x)$, $f'(x) < 0$ decreas $\forall x$

Example: Determine whether the following seq Monotone or not

1.) $\{\tan^{-1} x\}_{n=1}^{\infty}$

$f(x) = \tan^{-1} x$, $\forall x \geq 1$

$f'(x) = \frac{1}{1+x^2} > 0$ $\forall x \geq 1 \therefore$ increasing $\therefore$ Monotone.

2.) $\{3 + \frac{1}{n}\}_{n=1}^{\infty}$

$f(x) = 3 + \frac{1}{x}$, $\forall x \geq 1$

$f'(x) = \frac{-1}{x^2} < 0$ $\forall x \geq 1 \therefore$ decreasing $\forall n \geq 1 \therefore$ Monotone.

3.) $\left\{\frac{(10)^n}{n!}\right\}_{n=1}^{\infty}$

$a_n = \frac{10^n}{n!}$, $a_{n+1} = \frac{10^{n+1}}{(n+1)!}$

$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}}{(n+1)!} \div \frac{10^n}{n!}$

$= \frac{10^{n+1}}{(n+1)!} * \frac{n!}{10^n}$

$= \frac{\cancel{10^n}.10.\cancel{n!}}{(n+1).\cancel{n!}.\cancel{10^n}} = \frac{10}{n+1}$

$n=1 \Rightarrow \frac{a_{n+1}}{a_n} = \frac{10}{1+1} > 1$

$n=2 \Rightarrow \frac{a_{n+1}}{a_n} = \frac{10}{1+2} > 1$

$n=3 \Rightarrow \frac{a_{n+1}}{a_n} = \frac{10}{1+3} > 1$

$n=9 \Rightarrow \frac{a_{n+1}}{a_n} = \frac{10}{1+9} = 1$

$n=10 \Rightarrow \frac{a_{n+1}}{a_n} = \frac{10}{1+10} < 1$

$n=11 \Rightarrow \frac{a_{n+1}}{a_n} = \frac{10}{1+11} < 1$

$n=1$ ——— >1 ——— $n=9$ ——— <1

$\therefore$ decreasing $\forall n \geq 9$

4.) $\{n^2+5n\}_{n=1}^{\infty}$

$f(x) = x^2+5x$, $f'(x) = 2x+5$

$\therefore$ increasing $\forall n \geq 1$ $\frac{-5}{2}$ 1 + +

5.) $\left\{\frac{5}{n+3}\right\}_{n=1}^{\infty}$

$f(x) = \frac{5}{x+3}$, $x \geq 1$, $f'(x) = \frac{-5}{(x+3)^2} < 0$

$\therefore$ decreasing $\forall n \geq 1$

6.) $\{(-1)^n\}_{n=1}^{\infty}$ not monotone "متذبذب"

$-1, 1, -1, 1$

7.) $\left\{\frac{n}{n+2}\right\}_{n=1}^{\infty}$ $f(x) = \frac{x}{x+2}$ $\forall x \geq 1$

$f'(x) = \frac{(x+2)-(x)}{(x+2)^2} = \frac{2}{(x+2)^2} > 0$

$\therefore$ increasing $\forall n \geq 1$

8.) $\{\ln n\}_{n=2}^{\infty}$ $f(x) = \ln x$, $x \geq 2$

$f'(x) = \frac{1}{x}$ 0   2   ++

$\therefore$ increasing $\forall n \geq 2$

9.) $\left\{\frac{n!}{4^n}\right\}_{n=1}^{\infty}$ $a_n = \frac{n!}{4^n}$, $a_{n+1} = \frac{(n+1)!}{4^{n+1}}$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{4^{n+1}} * \frac{4^n}{n!} = \frac{n+1}{4}$

$n=1 \Rightarrow \frac{a_{n+1}}{a_n} = \frac{2}{4} < 1$

$n=3 \Rightarrow \frac{a_{n+1}}{a_n} = \frac{4}{4} = 1$

$n=4 \Rightarrow \frac{a_{n+1}}{a_n} = \frac{5}{4} > 1$

$\therefore$ increasing $\forall n \geq 3$

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Let $\{a_n\}_{n=1}^{\infty}$ be any seq. then the correspondening series $a_1+a_2+a_3 \dots = \sum_{k=1}^{\infty} a_k$

Now to find the sum of this series.

Let $S_1 = a_1$

$S_2 = a_1 + a_2 = S_1 + a_2$

$S_3 = a_1 + a_2 + a_3 = S_2 + a_3$

$\vdots$

$S_k = S_{k-1} + a_k$

Now.. $\{S_1, S_2, \dots, S_k\}$ $(S_k)_{k=1}^{\infty}$ seq of partial sums.

* if $\lim_{k\to\infty} S_k = L$ then $\sum_{k=1}^{\infty} a_k$ conv of sum $= \lim_{k\to\infty} S_k$
* if $\lim_{k\to\infty} S_k = \text{dne} \Rightarrow \sum_{k=1}^{\infty} a_k$ div of has no sum.

Examples: determine whether the following series conv or div? if it's conv find the sum?

1.) $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$, $S_1 = 1 - \frac{1}{2}$

$S_2 = \underbrace{(1-\frac{1}{2})}_{a_1} + \overbrace{(\frac{1}{2} - \frac{1}{3})}^{a_2} = 1 - \frac{1}{3}$

$S_3 = \underbrace{(1-\frac{1}{3})}_{a_1+a_2} + \underbrace{(\frac{1}{3} - \frac{1}{4})}_{a_3} = 1 - \frac{1}{4}$

$\vdots$

$S_k = 1 - \frac{1}{k+1}$, $\lim_{k\to\infty} 1 - \frac{1}{k+1} = 1 \therefore$ conv

$\sum = 1$

2.) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

$= \frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$\therefore 1 = A(n+1) + B(n)$ if $n=0 \Rightarrow A = 1$

if $n=-1 \Rightarrow B = -1$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum \frac{1}{n} - \frac{1}{n+1}$ نفس السؤال السابق

$= 1$

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3-) $\sum_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2+k}} = \sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{\sqrt{k(k+1)}} - \frac{\sqrt{k}}{\sqrt{k(k+1)}}$

$= \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$

$S_1 = 1 - \frac{1}{\sqrt{2}}$

$S_2 = \underbrace{\left(1-\frac{1}{\sqrt{2}}\right)}_{a_1} + \underbrace{\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)}_{a_2} = 1 - \frac{1}{\sqrt{3}}$

$S_3 = \underbrace{\left(1-\frac{1}{\sqrt{3}}\right)}_{a_1+a_2} + \left(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}\right) = 1 - \frac{1}{\sqrt{4}}$ $\therefore S_k = 1 - \frac{1}{\sqrt{k+1}}$

Now, $\lim_{k\to\infty} S_k = \lim_{k\to\infty} 1 - \frac{1}{\sqrt{k+1}} = 1$

$\therefore \sum \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2+k}}$ conv of $\sum_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2+k}} = 1$

4-) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

$= \sum_{n=1}^{\infty} \ln(n+1) - \ln(n)$

1-) $S_1 = \ln 2 - \ln 1 = \ln 2$

2-) $S_2 = \cancel{\ln 1} + \underbrace{(\ln 3 - \ln 2)}_{a_2} = \ln 2 + \ln 3 - \ln 2 = \ln 3$

3-) $S_3 = \underbrace{\ln 3}_{(a_1+a_2)} + \underbrace{(\ln 4 - \ln 3)}_{a_3} = \ln 4$

$\therefore S_k = \ln(k+1)$

Now, $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \ln(k+1) = \infty$ $\therefore$ div of has no sum lecture 16: Series part 1 — 20/5/2022

$\boxed{11.2}$

5.) $\sum_{k=1}^{\infty} \tan^{-1}(k+1) - \tan^{-1}(k)$

Sol: $S_1 = \tan^{-1}(2) - \tan^{-1}(1)$

$S_2 = (\tan^{-1}(2) - \tan^{-1}(1)) + (\tan^{-1}(3) - \tan^{-1}(2)) = \tan^{-1}(3) - \tan^{-1}(1)$

$S_3 = (\tan^{-1}(3) - \tan^{-1}(1)) + (\tan^{-1}(4) - \tan^{-1}(3)) = \tan^{-1}(4) - \tan^{-1}(1)$

$\therefore S_k = \tan^{-1}(k+1) - \tan^{-1}(1)$, $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \tan^{-1}(k+1) - \tan^{-1}(1)$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ #

Example: if the series $\sum_{k=1}^{\infty} a_k$ has the n-th partial sum

$S_n = 2 + \frac{3n}{n+1}$ then $\sum_{k=1}^{\infty} a_k =$

$* \sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 + \frac{3n}{n+1} = 2 + 3 = 5$

H.W: determine whether the following series conv or div? if its conv find it's sum.

1.) $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right) - \sin\left(\frac{1}{k+1}\right)$

2.) $\sum_{k=3}^{\infty} \frac{1}{k^2-1}$

3.) $\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+2}$

4.) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right) - \cos\left(\frac{1}{(n+1)^2}\right)$

5.) $\sum_{k=1}^{\infty} \frac{1}{2^k} - \frac{1}{2^{k+1}}$

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* Geometric series: $\sum_{k=1}^{\infty} (r)^k = r + r^2 + r^3 + r^4 + \dots$

eg: $\sum_{k=1}^{\infty} 2^k$ Geometric

$\sum_{k=1}^{\infty} \frac{1}{4+k}$ not Geometric

$\sum_{k=1}^{\infty} \frac{1}{3^k} = \sum \left(\frac{1}{3}\right)^k$ Geometric

$\sum_{k=1}^{\infty} k^k$ not Geometric, $\sum_{k=1}^{\infty} k^3$ not Geometric

• Note that:

* $\sum_{k=1}^{\infty} r^k$ conv if $|r|<1$, $-1<r<1$ & has sum ⟹

$$\sum_{k=m}^{\infty} r^k = \frac{r^m}{1-r}$$

* $\sum_{k=1}^{\infty} r^k$ div if $|r| \geq 1$ and has no sum.

Examples: Determine whether the following series converge or diverge if it's conv find it's sum?

1-) $\sum_{k=0}^{\infty} \left(\frac{-1}{5}\right)^k$, Geometric series $\left|\frac{-1}{5}\right| = \frac{1}{5} < 1$ ∴ conv

$$\text{Sum} = \frac{\left(\frac{-1}{5}\right)^0}{1 - -\frac{1}{5}} = \frac{1}{1+\frac{1}{5}} = \frac{5}{6}$$

2-) $\sum_{k=0}^{\infty} \left(\frac{2}{e}\right)^k$, Geometric series $\left|\frac{2}{e}\right| < 1$ ∴ conv

$$\text{Sum} = \frac{\left(\frac{2}{e}\right)^0}{1-\frac{2}{e}} = \frac{1}{\frac{e-2}{e}} = \frac{e}{e-2}$$

3-) $\sum_{k=1}^{\infty} 2^{\frac{-k}{2}} = \sum_{k=1}^{\infty} \left(\frac{1}{2^{1/2}}\right)^k = \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^k$ Geometric $\left|\frac{1}{\sqrt{2}}\right| < 1$ ∴ conv

$$\text{sum} = \frac{\left(\frac{1}{\sqrt{2}}\right)^1}{1-\frac{1}{\sqrt{2}}} = \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{1}{\sqrt{2}-1} \#$$

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4-) $\sum_{k=0}^{\infty} e^{-k} = \sum_{k=0}^{\infty} (\frac{1}{e})^k$, $|\frac{1}{e}|<1$ $\therefore$ conv

$$\text{Sum} = \frac{(1/e)^0}{1-\frac{1}{e}} = \frac{1}{\frac{e-1}{e}} = \frac{e}{e-1}$$

5-) $\sum_{n=2}^{\infty} (\frac{-1}{3})^{n+1} = \sum_{n=2}^{\infty} (\frac{-1}{3})^n \cdot \frac{-1}{3}$

$= \frac{-1}{3} \sum_{n=2}^{\infty} (\frac{-1}{3})^n$, $|\frac{-1}{3}|<1$ $\therefore$ conv

$= \frac{-1}{3} \cdot \frac{(\frac{-1}{3})^2}{1-(\frac{-1}{3})} = \frac{-1}{3} \cdot \frac{1/9}{\frac{4}{3}} = \frac{-1}{36}$

Note that: if $\sum u_k$ conv then $\sum_{k=1}^{\infty} a\, u_k$ conv

of $\underbrace{\sum_{k=1}^{\infty} a u_k}_{(au_1+au_2+au_3+\dots)} = a \sum_{k=1}^{\infty} u_k \longrightarrow a(u_1+u_2+u_3+\dots)$

6-) $\sum_{k=1}^{\infty} (5)^{2k} (7)^{1-k} = \sum_{k=1}^{\infty} (5^2)^k \cdot \frac{7}{7^k}$

$= 7 \sum_{k=1}^{\infty} (\frac{25}{7})^k$, $|\frac{25}{7}|>1$ $\therefore$ div has no sum.

7-) $\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} = \sum_{n=0}^{\infty} (\frac{-1}{3})^n$, $|\frac{-1}{3}|<1$ $\therefore$ conv

$$\text{Sum} = \frac{(-1/3)^0}{1-\frac{-1}{3}} = \frac{1}{4/3} = \frac{3}{4}$$

8-) $\sum_{n=1}^{\infty} e^n \pi^{1-n} = \sum_{n=1}^{\infty} \frac{e^n \cdot \pi}{\pi^n} = \pi \sum_{n=1}^{\infty} (\frac{e}{\pi})^n$, $|\frac{e}{\pi}|<1$ $\therefore$ conv

$$\text{Sum} = \frac{(e/\pi)^1}{1-\frac{e}{\pi}} = \cancel{\pi} \cdot \frac{e}{\cancel{\pi}(\pi-e)} = \frac{\pi e}{\pi - e}$$

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9-) $\sum_{n=1}^{\infty} 3^{2n} = \sum_{n=1}^{\infty} 9^n$, $|9|>1$ ∴ div

10-) $\sum_{k=2}^{\infty} 3^{-k+2} = \sum_{k=2}^{\infty} \frac{3^2}{3^k} = 9\sum_{k=2}^{\infty} (\frac{1}{3})^k$, $|\frac{1}{3}|<1$ ∴ conv

$$\text{Sum} = \frac{9 \cdot (1/3)^2}{1-\frac{1}{3}} = \frac{3}{2}$$

11-) $\sum_{k=1}^{\infty} \frac{4^{-k+1}}{3^{-2k+2}} = \sum_{k=1}^{\infty} \frac{4 \cdot 4^{-k}}{9 \cdot 3^{-2k}} = \frac{4}{9}\sum_{k=1}^{\infty} (\frac{3^2}{4})^k = \frac{4}{9}\sum_{k=1}^{\infty} (\frac{9}{4})^k$, $|\frac{9}{4}|>1$ ∴ div

12-) $\sum_{k=3}^{\infty} \frac{(-2)^{k+1}}{4^{\frac{1}{2}k+2}} = \sum_{k=3}^{\infty} \frac{-2 \cdot (-2)^k}{(4)^2 \cdot ((4)^{1/2})^k} = \frac{-2}{16}\sum_{k=3}^{\infty} (\frac{-2}{2})^k$ $|-1|=1$ ∴ div

• Note that: if $\sum a_k$ conv, $\sum b_k$ conv

then $\sum a_k + b_k$ conv of $\sum a_k + b_k = \sum a_k + \sum b_k$

13-) $\sum_{k=1}^{\infty} (\frac{1}{\pi})^k + (\frac{3}{\pi})^k$ → conv $\frac{3}{\pi}<1$

conv $\frac{1}{\pi}<1$

$$\therefore = \sum_{k=1}^{\infty} (\frac{1}{\pi})^k + \sum_{k=1}^{\infty} (\frac{3}{\pi})^k$$

$$\text{Sum} = \frac{(1/\pi)^1}{1-\frac{1}{\pi}} + \frac{(3/\pi)^1}{1-\frac{3}{\pi}} \text{ - - - - -}$$

14-) $\sum_{k=1}^{\infty} (\frac{2}{3})^{k+2} = (\frac{2}{3})^2 \sum_{k=1}^{\infty} (\frac{2}{3})^k$, $|\frac{2}{3}|<1$

$$\text{Sum} = \frac{4}{9} \cdot \frac{(2/3)^1}{1-\frac{2}{3}} = \text{- -}$$

or $\sum_{k=1}^{\infty} (\frac{2}{3})^{k+2} = \sum_{k=1+2}^{\infty} (\frac{2}{3})^{k+2-2} = \underbrace{\sum_{k=3}^{\infty} (\frac{2}{3})^k}_{(\frac{2}{3})^3 + (\frac{2}{3})^4 + (\frac{2}{3})^5 + \dots}$

$(\frac{2}{3})^3 + (\frac{2}{3})^4 + \dots$

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15-) $\sum_{k=1}^{\infty} (-1)^{k-1} \cdot \frac{7}{6^{k-1}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(6)^k} \cdot 7$

$= 7 \cdot \frac{(-1/6)^0}{1 - -\frac{1}{6}} = 6$

16-) $\sum_{k=1}^{\infty} \frac{(4)^{k+2}}{(7)^{k-1}} = \frac{(4)^2}{(7)^{-1}} \cdot \sum_{k=1}^{\infty} (\frac{4}{7})^k$

$= 16 \cdot 7 \sum_{k=1}^{\infty} (\frac{4}{7})^k = 16 \cdot 7 \cdot \frac{4/7}{1 - 4/7} = ---$

Example: if $\sum_{k=2}^{\infty} 2r^k = 1$ then $r = ??$

$2 \sum_{k=2}^{\infty} r^k = 1 \Rightarrow \sum_{k=2}^{\infty} r^k = \frac{1}{2}$

$\frac{r^2}{1-r} = \frac{1}{2}$

$2r^2 = 1 - r$

$2r^2 + r - 1 = 0$

$(2r-1)(r+1) = 0$

$r = \frac{1}{2}$ $\quad$ $\hookrightarrow r = -1$ div

Example: find all values of r such that $\sum_{k=1}^{\infty} \frac{1}{r^k}$ conv

$\sum \frac{1}{r^k}$ conv $\Rightarrow \sum (\frac{1}{r})^k$

$\therefore |\frac{1}{r}| < 1$ $\qquad -1 < \frac{1}{r} < 1$

$-1 > r > 1$

$\therefore (-\infty, -1) \cup (1, \infty)$

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* Divergent test: "أول تيست منجربه دايماً"

• Thrm: if $\sum_{k=1}^{\infty} a_k$ conv then $\lim_{k\to\infty} a_k = 0$

• Div test: if $\lim_{k\to\infty} a_k \neq 0 \Rightarrow \sum a_k$ div (قاعدة بتحكم بالتباعد فقط)

e.g: $\sum_{k=1}^{\infty} (\frac{1}{3})^k$, Geomtric series, $\frac{1}{3} < 1$ $\therefore$ conv
conv or div? $\lim_{k\to\infty} (\frac{1}{3})^k = 0$ $\therefore$ faild

e.g: $\sum_{k=1}^{\infty} \frac{3-2k^2}{4+k+7k^2}$ conv or div? $\lim_{k\to\infty} \frac{3-2k^2}{4+k+7k^2} = \frac{-2}{7} \neq 0$

$\therefore$ div by div test

e.g: $\sum_{k=1}^{\infty} (1-\frac{2}{k})^{3k}$, $\lim_{k\to\infty} (1-\frac{2}{k})^{3k} = (e^{-2})^3 \neq 0$ $\therefore$ div by div test.
~~conv or div?~~

e.g: $\sum_{k=1}^{\infty} (\frac{3k}{3k+1})^k$, $\lim_{k\to\infty} (\frac{3k}{3k+1})^k$
conv or div?

$= \lim_{k\to\infty} (\frac{3k+1}{3k})^{-k}$

$= \lim_{k\to\infty} (1+\frac{1}{3k})^{-k}$

$= (e^{1/3})^{-1} = \frac{1}{\sqrt[3]{e}} \neq 0$ $\therefore$ div by div test.

e.g: $\sum_{k=1}^{\infty} \frac{1}{k^2+9}$ conv or div?

$= \lim_{k\to\infty} \frac{1}{k^2+9} = 0$ $\therefore$ div test failed

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e.g: $\sum_{n=2}^{\infty} \frac{n}{\ln n}$ conv or div?

$= \lim_{n\to\infty} \frac{n}{\ln n} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty \neq 0$ $\therefore$ div by div test.

e.g: $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{2n^2+1}$

A.S $\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2} \neq 0$ $\therefore$ div by div test.

H.W: determine whether the following series conv or div?

1.) $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{k}-1}$

2.) $\sum_{k=1}^{\infty} \frac{e^k+7}{k^2+3k+1}$

3.) $\sum_{k=1}^{\infty} \frac{2^k+1}{3^k}$

4.) $\sum_{k=1}^{\infty} k^2 \sin^2\left(\frac{1}{k}\right)$

5.) $\sum_{k=1}^{\infty} \frac{1}{2+3^{-k}}$

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* Integral test: "إذا سهل تكامله"

For $\sum_{n=1}^{\infty} a_n$, let $a_n = f(x)$, $x \geq 1$

① if $f(x) > 0 \ \forall x \geq 1$

② $f(x)$ decreasing $\forall x \geq 1$

③ $f(x)$ cont $\forall x \geq 1$

then $\int_1^{\infty} f(x)\,dx \begin{cases} \to \text{conv} \Rightarrow \sum a_n \text{ conv} \\ \to \text{div} \Rightarrow \sum a_n \text{ div} \end{cases}$

Example: $\sum_{k=3}^{\infty} \frac{\ln k}{k}$ conv or div ?

Div test:

1-) $\lim_{k\to\infty} \frac{\ln k}{k} \overset{LH}{=} \lim_{k\to\infty} \frac{1/k}{1} = 0 \ \therefore$ div test failed.

2-) Integral test

$f(x) = \frac{\ln x}{x}, \ \forall x \geq 3$

① $f(x) > 0 \ \forall x \geq 3$

② $f'(x) = \frac{x * \frac{1}{x} - \ln x}{x^2} = \frac{1-\ln x}{x^2}$ — e — 3 $\therefore$ dec $\forall x \geq 3$

③ $f(x)$ cont $(0,\infty)$ $\therefore$ cont $\forall x \geq 3$

Now $\int_3^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_3^{b} \frac{\ln x}{x}$

$= \lim_{b\to\infty} \frac{(\ln x)^2}{2} \Big]_3^{b} = \infty \ \therefore$ div

$\therefore$ div by integral test.

$u = \ln x$

$du = \frac{dx}{x}$

$\int \frac{\ln x}{x} = \int u\,du$

$= \frac{(\ln x)^2}{2}$

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Example: $\sum_{K=1}^{\infty} K e^{-K}$ conv or div?

Div test:

① $\lim_{K\to\infty} K e^{-K} = \lim_{K\to\infty} \frac{K}{e^{K}} \overset{L'H}{=} \lim_{K\to\infty} \frac{1}{e^{K}} = 0$ ∴ div test faild.

② integral test:

$f(x) = x e^{-x}, \forall x \geq 1$

① $f(x) > 0, \forall x \geq 1$

② $f'(x) = -x e^{-x} + e^{-x} = e^{-x}[-x+1]$

∴ dec $\forall x \geq 1$

③ $f(x)$ cont $\forall x \geq 1$

Now.. $\int_1^{\infty} x e^{-x} dx = \lim_{b\to\infty} \int_1^{b} x e^{-x}$

u	dv
x	e^{-x}
1	$-e^{-x}$
0	$+e^{-x}$

$= \lim_{b\to\infty} \left[-x e^{-x} - e^{-x}\right]_1^{b}$

$= \lim_{b\to\infty} -b e^{-b} - 0 - [-e^{-1} - e^{-1}]$

$= \lim_{b\to\infty} \frac{-b}{e^{b}} + 2e^{-1} = 2e^{-1}$ ∴ conv

(zero)

∴ $\sum_{K=1}^{\infty} K e^{-K}$ conv by integral test.

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Example: $\sum_{k=3}^{\infty} \frac{1}{k\ln k}$ conv or div?

1-) $\lim_{k\to\infty} \frac{1}{k\ln k} = 0$ ∴ div test failed.

2-) integral test

$f(x) = \frac{1}{x\ln x}$, $x \geq 3$

① $f(x) > 0 \quad \forall x \geq 3$

② $f'(x) = \frac{-(x \cdot \frac{1}{x} + \ln x)}{(x\ln x)^2} = \frac{-1-\ln x}{(x\ln x)^2}$ $\quad e \quad 3$

decreasing $\forall x \geq 3$

③ $f(x)$ cont $(1,\infty)$ ∴ cont $\forall x \geq 3$

Now.. $\int_3^{\infty} \frac{1}{x\ln x} = \lim_{b\to\infty} \ln(\ln x)\Big|_3^b$

$= \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 3)]$

$= \infty$ ∴ div

∴ $\sum \frac{1}{k\ln k}$ div by integral test

H.W: 1-) $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k^2+1}$ conv or div?

2) $\sum_{k=1}^{\infty} k^2 e^{-k^3}$ conv or div?

lecture 18: Series part 3

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Example: $\sum_{k=3}^{\infty} \frac{1}{k^2+9}$ conv or div?

① div test: $\lim_{k\to\infty} \frac{1}{k^2+9} = 0$ ∴ faild.

② integral test: $f(x) = \frac{1}{x^2+9}$, $\forall x \geq 3$

① $f(x) > 0 \quad \forall x \geq 3$

② $f'(x) = \frac{-2x}{x^2+9}$ --- dec $\forall x \geq 3$

③ $f(x)$ cont $\forall x \geq 3$

Now.. $\int_3^{\infty} \frac{1}{x^2+9}\,dx = \lim_{t\to\infty} \int_3^t \frac{dx}{x^2+9}$

$= \lim_{t\to\infty} \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right)\Big|_3^t$

$\lim_{t\to\infty} \frac{1}{3}\left[\tan^{-1}\left(\frac{t}{3}\right) - \tan^{-1}\left(\frac{3}{3}\right)\right]$

$= \frac{1}{3}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$

$= \frac{1}{3} * \frac{\pi}{4} = \pi/12$ ∴ conv

∴ $\sum_{k=3}^{\infty} \frac{1}{k^2+9}$ conv by integral test.

H.W: $\sum_{n=1}^{\infty} \frac{1}{n+1}$ conv or div

lecture 19: Series part 4 24/5/2022

- $\boxed{\text{p-series test}}$

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = \begin{cases} p>1 : \text{conv} \\ 0 \le p \le 1 : \text{div} \end{cases}$$

* if $p=1$, $\sum \frac{1}{k}$ div of called Harmonic series

(e.g) $\sum \frac{1}{k^3}$ conv or div?

$p = 3 > 1 \therefore$ conv

(e.g) $\sum \frac{1}{\sqrt[3]{k}}$ conv or div?

$= \sum \frac{1}{k^{1/3}}$, $p = \frac{1}{3} < 1 \therefore$ div

(e.g) $\sum k^{-5}$ conv or div

$= \sum \frac{1}{k^5}$, $p = 5 > 1 \therefore$ conv

(e.g) $\sum \frac{1}{\sqrt{n}}$ conv or div?

$= \sum \frac{1}{n^{1/2}}$, $p = \frac{1}{2} < 1 \therefore$ div

(e.g) $\sum_{k=1}^{\infty} \frac{k}{k^e}$ conv or div

$= \sum \frac{1}{k^{e-1}}$, $p = e-1 > 1 \therefore$ conv

(e.g) $\sum \frac{3}{4k}$ conv or div

$= \frac{3}{4} \sum \frac{1}{k}$, $p=1$ div

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lecture 19: Series part 4

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(e.g) $\sum \left(\frac{2}{K}\right)^4$ conv or div?

$= 2^4 \sum \frac{1}{K^4}$, $p = 4 > 1$ $\therefore$ conv

(e.g) $\sum_{K=1}^{\infty} \frac{1}{\sqrt{K+1}} = \sum_{K=1+1} \frac{1}{\sqrt{K+1-1}} = \sum_{K=2} \frac{1}{\sqrt{K}}$

$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}}$ ✓ $\qquad \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}}$ ✓

(e.g) $\sum_{K=3}^{\infty} \frac{1}{(K-2)^2} = \sum_{K=3-2}^{\infty} \frac{1}{(K-2+2)^2} = \sum_{K=1}^{\infty} \frac{1}{K^2}$, $p = 2 > 1$ $\therefore$ conv

Example: determine whether the following series conv or div?

1-) $\sum_{K=1}^{\infty} e^{-K}$: Geometric series

$= \sum_{K=1}^{\infty} \left(\frac{1}{e}\right)^K$ $\left|\frac{1}{e}\right| < 1$ $\therefore$ conv by Geometric series test

(r أو 1)

2-) $\sum_{K=1}^{\infty} K^{-e} = \sum_{K=1}^{\infty} \frac{1}{K^e}$: $p = 2 > 1$ $\therefore$ conv by p-series

3-) $\sum_{K=1}^{\infty} e^{K}$. Geometric series $|e| > 1$

4-) $\sum_{K=1}^{\infty} \frac{1}{e^{1/K}}$ not Geometric series
not p-series

هو مش مع الستاذ

div test : $\lim_{K \to \infty} \frac{1}{e^{1/K}} = 1 \neq 0$ $\therefore$ div by div test

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lecture 19: Series part 4 | 24/5/202

Limit comparision test

$\frac{\text{power}}{\text{power}}$

$\Rightarrow$ Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms

a-) choose $\sum_{k=1}^{\infty} b_k \rightarrow \frac{\text{أعلى أس}}{\text{أعلى أس}}$ اهم اشي

b-) find $\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$

if $\rho > 0$ then: if $\sum_{k=1}^{\infty} b_k$ conv $\Rightarrow \sum_{k=1}^{\infty} a_k$ conv

if $\sum_{k=1}^{\infty} b_k$ div $\Rightarrow \sum a_k$ div

Example: $\sum_{k=1}^{\infty} \underbrace{\frac{1}{k^4+1}}_{a_k}$ conv or div

① $\sum b_k = \sum_{k=1}^{\infty} \frac{1}{k^4}$

② $\rho = \lim_{k\to\infty} \frac{1}{k^4+1} * k^4 = 1 > 0$

Now .. $\sum b_k = \sum_{k=1}^{\infty} \frac{1}{k^4}$, $p = 4 > 1$, conv

$\therefore \sum_{k=1}^{\infty} \frac{1}{k^4+1}$ conv by L.C.T

Example: $\sum \underbrace{\frac{2k^2+7}{10k^3+16k^2+9}}_{a_k}$ conv or div

① $\sum b_k = \sum \frac{2k^2}{10k^3} = \sum \frac{1}{5k}$

② $\rho = \lim_{k\to\infty} \frac{2k^2+7}{10k^3+16k^2+9} * \frac{5k}{1} = 1 > 0$

$\therefore \sum b_k = \sum \frac{1}{5k}$ div $p=1$

$\sum a_k = \sum \frac{2k^2}{10k^3+16k^2+9}$ div by L.C.T.

lecture 14: Series part 4 25/5/2022

Example: $\sum_{k=1}^{\infty} \underbrace{\frac{k^{4/3}}{k^2+3k+4}}_{a_k}$ conv, div ?

1-) $\sum b_k = \sum \frac{k^{4/3}}{k^2} = \sum \frac{1}{k^{2/3}}$

2-) $\rho = \lim_{k\to\infty} \frac{k^{4/3}}{k^2+3k+4} \cdot \frac{k^{2/3}}{1} = 1 > 0$

$\therefore \sum b_k = \sum \frac{1}{k^{2/3}}$, $p = 2/3 < 1$ $\therefore$ div

$\sum a_k = \sum \frac{k^{4/3}}{k^2+3k+4}$ div by L.C.T

Example: $\sum_{k=1}^{\infty} \frac{k}{(2k+1)^{15}}$ conv or div

1-) $\sum b_k = \sum \frac{k}{k^{15}} = \sum \frac{1}{k^{14}}$

2-) $\underset{(a_k)}{\rho} = \lim_{k\to\infty} \frac{k}{(2k+1)^{15}} \cdot k^{14} = \frac{1}{2^{15}} > 0$

$\therefore \sum b_k = \sum \frac{1}{k^{14}}$, $p = 14 > 1$ $\therefore$ conv

$\therefore \sum a_k = \sum \frac{k}{(2k+1)^{15}}$ conv by L.C.T

Example: $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{(2k+1)^2}$ conv or div ?

1-) $\sum b_k = \sum \frac{k^{1/2}}{k^2} = \sum \frac{k^{3/2}}{1}$

2-) $\rho = \lim_{k\to\infty} \sum \frac{\sqrt{k}}{(2k+1)^2} * \frac{k^{3/2}}{1} = \frac{1}{4} > 0$

$\therefore \sum b_k = \sum \frac{1}{k^{3/2}}$, $p = 3/2 > 1$ $\therefore$ conv

$\therefore \sum a_k = \sum \frac{\sqrt{k}}{(2k+1)^2}$ conv by L.C.T

H.W: 1-) $\sum_{k=1}^{\infty} \frac{3}{k^{5/2}+1}$ 2-) $\sum_{k=1}^{\infty} \frac{n}{n^{3/2}-\frac{1}{2}}$

Lecture 20: Series part 5 25/5/2022

• Alternating series test (A.S.T)

$\sum (-1)^k a_k$, $\sum (-1)^{k+1} a_k$ $\sum \cos(\pi k) a_k$

*If 1-) $a_k > 0$ 2-) $\lim_{k\to\infty} a_k = 0$ 3-) a_k decreasing

then the series conv

*If $\lim_{k\to\infty} a_k \neq 0$ ∴ div by div test

Example: $\sum_{k=1}^{\infty} (-1)^{k+1} \underbrace{\frac{k+3}{k(k+1)}}_{a_k}$ conv or div

1-) $a_k = \frac{k+3}{k(k+1)} > 0$

2-) $\lim_{k\to\infty} \frac{k+3}{k(k+1)} = 0$

3-) $(a_k)' = \frac{k(k+1) - (k+3)(2k+1)}{(k^2+k)^2} = \frac{k^2+k-2k^2-7k-3}{(k^2+k)^2} = \frac{-(k^2+6k+3)}{(k^2+k)^2} < 0$

∴ $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$ conv by A.S.T

Example: $\sum (-1)^k \frac{k^2+1}{k+3}$, $\lim_{k\to\infty} \frac{k^2+1}{k+3} = \infty \neq 0$

∴ div by div test

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lecture 20: Series part 5

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Def: 1-) $\sum a_k$ conv absolutely if $\sum |a_k|$ conv
2-) $\sum a_k$ div absolutely if $\sum |a_k|$ div

Thrm: 1-) if $\sum a_k$ conv abs $\Rightarrow \sum a_k$ conv $\quad p \to q$
2-) if $\sum a_k$ div $\Rightarrow \sum a_k$ div abs $\quad \neg q \to \neg p$

Note that: if $\sum a_k$ div abs + $\sum a_k$ conv by A.S.T $\Rightarrow \sum a_k$ conv conditionally

Examples: $\sum \frac{(-1)^k}{k^3}$

1-) $\sum \left|\frac{(-1)^k}{k^3}\right| = \sum \frac{1}{k^3}$, $p=3>1 \therefore$ conv abs $\Rightarrow$ conv

Example: $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$

1-) $\sum \left|\frac{(-1)^k}{k}\right| = \sum \frac{1}{k}$, $p=1 \therefore$ div abs

2-) $a_k = \frac{1}{k}$ by A.S.T

* $a_k > 0$ * $\lim_{k\to\infty} \frac{1}{k} = 0$ * $(a_k)' = \frac{-1}{k^2} < 0$ decreasing

$\therefore$ conv by A.S.T

lecture 20: Series part 5 25/5/2022

Examples: $\sum_{k=1}^{\infty} \frac{k\cos(\pi k)}{k^2+1}$

1-) A.S : $\sum_{k=1}^{\infty} \left| \frac{k\cos(\pi k)}{k^2+1} \right| = \sum_{k=1}^{\infty} \frac{k}{k^2+1}$

* Now by L.C.T $\Rightarrow b_k = \frac{k}{k^2} = \frac{1}{k}$

* $\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k}{k^2+1} * \frac{k}{1} = \lim_{k\to\infty} \frac{k^2}{k^2+1} = 1 > 0$

$\therefore \sum \frac{1}{k}$ div $\Rightarrow \sum_{k=1}^{\infty} \frac{k}{k^2+1}$ div

$\therefore \sum \frac{k\cos(\pi k)}{k^2+1}$ div abs

Now.. A.S.T

1-) $a_k = \frac{k}{k^2+1} > 0$

2-) $\lim_{k\to\infty} \frac{k}{k^2+1} = 0$

3-) $(a_k)' = \frac{(k^2+1) - (k)(2k)}{(k^2+1)^2} = \frac{-k^2+1}{(k^2+1)^2} < 0$

$\therefore$ dec $\therefore$ conv by A.S.T

conv by A.S.T + div abs $\Rightarrow \sum$ conv conditionally

H.W: 1-) $\sum_{k=4}^{\infty} \frac{(-1)^k}{\sqrt[3]{2k^2-3k}}$ 2-) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}$ 3-) $\sum_{k=3}^{\infty} \frac{(-1)^k \ln x}{k}$

4-) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)(n+1)}$ 5-) $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n^3}$

lecture 21: Series part 6 1/6/2022

- Ratio test (! + ( )^K)

$\rho = \lim_{K\to\infty} \left|\frac{a_{K+1}}{a_K}\right|$ if: 1-) $\rho < 1$ conv abs
2-) $\rho > 1$ div
3-) $\rho = 1$ test failed

- Root test ( )^K

$\rho = \lim_{K\to\infty} (|a_K|)^{1/K}$ if: 1-) $\rho < 1$ conv abs
2-) $\rho > 1$ div
3-) $\rho = 1$ test failed.

Example: $\sum_{K=1}^{\infty} \frac{(-1)^K (2)^K}{K!}$

$\rho = \lim_{K\to\infty} \left|\frac{a_{K+1}}{a_K}\right| = \lim_{K\to\infty} \frac{(2)^{K+1}}{(K+1)!} \cdot \frac{K!}{2^K} = \lim_{K\to\infty} \frac{\cancel{2^K} \cdot 2}{(K+1)\cancel{K!}} \cdot \frac{\cancel{K!}}{\cancel{2^K}} = \lim_{K\to\infty} \frac{2}{K+1} = 0 < 1$

$\therefore$ conv abs by ratio test.

Example: $\sum_{n=1}^{\infty} \left(\frac{2n-1}{5n+10}\right)^n$

$\rho = \lim_{K\to\infty} \left(\left(\frac{2n-1}{5n+10}\right)^n\right)^{\frac{1}{n}} = \lim_{K\to\infty} \frac{2n-1}{5n+10} = \frac{2}{5} < 1$

$\therefore$ conv by root test.

Example: $\sum \frac{e^{2K}}{K^{2K}}$

$= \sum \left(\frac{e^2}{K^2}\right)^K$, $\rho = \lim_{K\to\infty} \left(\left(\frac{e^2}{K^2}\right)^K\right)^{1/K} = \lim_{K\to\infty} \frac{e^2}{K^2} = 0 < 1$

$\therefore$ conv by root test.

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lecture 21: Series part 6 1/6/2022

Example: $\sum \frac{1}{K!}$

$$\rho = \lim_{K\to\infty} \frac{1}{(K+1)!} \cdot K! = \lim_{K\to\infty} \frac{\cancel{K!}}{(K+1)\cancel{K!}} = \lim_{K\to\infty} \frac{1}{K+1} = 0 < 1$$

$\therefore$ conv by ratio test.

Example: $\sum_{n=1}^{\infty} \left(\frac{n+2}{n}\right)^{n^2}$

$$\rho = \lim_{n\to\infty} \left(\left(\frac{n+2}{n}\right)^{n^2}\right)^{1/n} = \lim_{K\to\infty} \left(\frac{n+2}{n}\right)^n = \lim_{K\to\infty} \left(1+\frac{2}{n}\right)^n = e^2 > 1$$

$\therefore$ div by root test.

Example: $\sum_{n=1}^{\infty} \left(1+\frac{3}{n}\right)^n$

$$\rho = \lim_{K\to\infty} \left(\left(1+\frac{3}{n}\right)^n\right)^{1/n} = \lim_{K\to\infty} \left(1+\frac{3}{n}\right) = 1$$ $\therefore$ root test failed

div test: $\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n = e^3 \neq 0$ $\therefore$ div by div test.

Example: $\sum K^{50} \cdot e^{-K}$

$$= \sum \frac{K^{50}}{e^K}$$

$$\rho = \lim_{K\to\infty} \frac{(K+1)^{50}}{e^{K+1}} \cdot \frac{e^K}{K^{50}} = \lim_{K\to\infty} \frac{1}{e} \cdot \left(\frac{K+1}{K}\right)^{50}$$

$$= \frac{1}{e}\left(\lim_{K\to\infty} \frac{K+1}{K}\right)^{50}$$

$$= \frac{1}{e} * 1 = \frac{1}{e} < 1$$ $\therefore$ conv by ratio test.

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Example: $\sum \left(\frac{n+1}{n+2}\right)^n$

1-) $\rho = \lim_{k\to\infty} \left(\left(\frac{n+1}{n+2}\right)^n\right)^{1/n} = \lim_{n\to\infty} \frac{n+1}{n+2} = 1$ ∴ root test failed.

2-) div test $= \lim_{k\to\infty} \left(\frac{n+1}{n+2}\right)^n$

$= \lim_{n\to\infty} \frac{\left(n\left(1+\frac{1}{n}\right)\right)^n}{\left(n\left(1+\frac{2}{n}\right)\right)^n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}$

$= \frac{e}{e^2} = \frac{1}{e} \neq 0$ ∴ div by div test

Example: $\sum \left(\frac{k+1}{3k+4}\right)^{2k}$

$\rho = \lim_{k\to\infty} \left(\left(\frac{k+1}{3k+4}\right)^{2k}\right)^{1/k}$

$= \lim_{k\to\infty} \left(\frac{k+1}{3k+4}\right)^2 = \frac{1}{9} < 1$ ∴ conv by root test.

Examples: $\sum \frac{(-1)^k k^k}{k!}$

$\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k}$

$= \lim_{k\to\infty} \frac{(k+1)^k \cdot \cancel{k+1}}{(\cancel{k+1})\cancel{k!}} \cdot \frac{\cancel{k!}}{k^k}$

$= \lim_{k\to\infty} \frac{(k+1)^k}{k^k}$

$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k$

$= \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e > 1$ ∴ div by ratio test.

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Example: $\sum \frac{(k!)^2}{(2k)!}$

$$\rho = \lim_{k \to \infty} \frac{((k+1)!)^2}{(2(k+1))!} * \frac{(2k)!}{(k!)^2}$$

$$= \lim_{k \to \infty} \frac{(k+1)^2 \cdot \cancel{(k!)^2}}{(2k+2)!} \cdot \frac{2k!}{\cancel{(k!)^2}}$$

$$= \lim_{k \to \infty} \frac{(k+1)^2 \cancel{(2k!)}}{(2k+2)(2k+1)\cancel{(2k)!}}$$

$$= \lim_{k \to \infty} \frac{(k+1)^2}{(2k+2)(2k+1)} = \frac{1}{4} < 1$$ $\therefore$ conv by ratio test.

lecture 22: power series

2/6/2022

$*\ \sum_{n=0}^{\infty} a_n (x-x_0)^n$

e.g.) $\sum_{n=0}^{\infty} x^n$ $\quad a_n = 1,\ x_0 = 0$

e.g.) $\sum_{n=0}^{\infty} n!\,x^n$ $\quad a_n = n!,\ x_0 = 0$

e.g.) $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}$ $\quad a_n = \frac{1}{n!},\ x_0 = 3$

e.g.) $\sum_{n=0}^{\infty} x^n$, find the value of x which make the series conv.

$|x| < 1 \ \therefore x \in (-1, 1)$ "interval of convergence"

$\frac{1-(-1)}{2} = \frac{2}{2} = 1$ "radius of conv"

Thrm: For any power series there exist radius of convergence and interval of convergence.

1-) How to find the radius of conv?

radius $= \rho = \frac{1}{L}$ where $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

2-) interval of conv?

$\Rightarrow |x - x_0| < \rho$

lecture 22: power series 2/6/2022

Example: find the radius of interval of convergence of :-

1-) $\sum_{K=0}^{\infty} \frac{(x+2)^K}{K \cdot 4^K}$, $a_K = \frac{1}{K 4^K}$, $x_0 = -2$

• to find the radius of conv :-

1-) $L = \lim_{K\to\infty} \frac{a_{K+1}}{a_K} = \lim_{K\to\infty} \frac{1}{(K+1) 4^{K+1}} \cdot K \cdot 4^K$

$= \lim_{K\to\infty} \frac{K \cdot \cancel{4^K}}{(K+1) \cdot \cancel{4^K} \cdot 4}$

$= \frac{1}{4} \lim_{K\to\infty} \frac{K}{K+1} = \frac{1}{4}$

2-) $\rho = \frac{1}{L} = 4$

• the interval of conv :-

$|x+2| < 4 \Rightarrow -4 < x+2 < 4$

*نشوف الاطراف؟

$-6 < x < 2$

1-) if $\boxed{x=-6}$, $\sum_{K=0}^{\infty} \frac{(-6+2)^K}{4^K K} = \sum_{K=0}^{\infty} \frac{-4^K}{4^K K} = \sum_{K=0}^{\infty} \frac{(-1)^K}{K}$

by A.S.T $\sum \frac{(-1)^K}{K}$ conv ∴ [-6,

2-) if $\boxed{x=2}$, $\sum \frac{(2+2)^K}{4^K K} = \sum \frac{4^K}{K 4^K} = \sum \frac{1}{K}$ ∴ div

∴ interval of conv [-6, 2)

Five Appl

lecture 22: power series 2/6/2022

Example: $\sum_{K=0}^{\infty} K!(x-5)^K$, $a_K = K!$, $x_0 = 5$

1-) radius of conv :-

$$L = \lim_{K\to\infty} \frac{(K+1)!}{K!} = \lim_{K\to\infty} (K+1) = \infty$$

$$\rho = \frac{1}{L} = \frac{1}{\infty} = 0$$

2-) $|x-5| < 0 \Rightarrow x = 5$ ∴ conv only at $x = 5$

Example: $\sum_{K=0}^{\infty} \frac{(-1)^K 10^K (x-1)^K}{K!}$, $a_K = \frac{(-1)^K 10^K}{K!}$

1-) radius of conv

$$L = \lim_{K\to\infty} \left|\frac{a_{K+1}}{a_K}\right| = \lim_{K\to\infty} \frac{10^{K+1}}{(K+1)!} \cdot \frac{K!}{10^K}$$

$$= \lim_{K\to\infty} \frac{10}{K+1} = 0$$

$$\therefore \rho = \frac{1}{L} = \frac{1}{0} = \infty$$

2-) interval of conv $(-\infty, \infty)$

Defin: Let $f(x)$ be diffble for all order at constant (x_0), then the taylor series for $f(x)$ is :-

$$f(x) = \sum_{K=0}^{\infty} \frac{f^{(K)}(x_0)}{K!}(x-x_0)^K$$

$$= f(x_0) + \frac{f'(x_0)(x-x_0)^1}{1!} + \frac{f^{(2)}(x_0)(x-x_0)^2}{2!} + $$

Five Apple

lecture 22: power series 2/6/2022

Example: Let $f(x) = \frac{1}{x+3}$, find the taylor series for $f(x)$ at $x_0 = 2$?

1-) $f(2) = \frac{1}{5}$

2-) $f'(2) = \frac{-1}{(x+3)^2}\Big|_{x=2} = \frac{-1}{25} = \frac{-1}{5^2}$

3-) $f''(2) = \frac{2(x+3)}{(x+3)^4} = \frac{2}{(x+3)^3} = \frac{2}{125} = \frac{2}{5^3}$

4-) $f^{(3)}(2) = 6(x+3)^{-4}\Big|_{x=2} = \frac{-6}{5^4}$

5-) $f^{(4)}(2) = 24(x+3)^{-5}\Big|_{x=2} = \frac{24}{5^5}$

$\therefore f^{(k)}(2) = \frac{(-1)^k k!}{5^{k+1}}$

lecture 23: Maclaurin Series 3/6/2022

Recall that power series $= \sum_{n=0}^{\infty} a_n (x - x_0)^n$

taylor series for $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$

Mac series for $f(x) = \sum_{k=0}^{\infty} \frac{f^{k}(0)}{k!} x^k$

Example: find the Maclaurin series for $f(x) = e^x$

1-) $f^{(0)}(x) = e^x \longrightarrow f^{(0)}(0) = 1$

2-) $f^{(1)}(x) = e^x \longrightarrow f^{(1)}(0) = 1$

3-) $f^{(2)}(x) = e^x \longrightarrow f^{(2)}(0) = 1$

$\vdots$

4-) $f^{(k)}(x) = e^x \longrightarrow f^{(k)}(0) = 1$

$$\therefore f(x) = e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- Some important Maclaurin Series.

1-) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $x \in \mathbb{R}$

2-) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, $x \in \mathbb{R}$

"زوجي"

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

3-) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $x \in \mathbb{R}$

"فردي"

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

4-) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$

$$= 1 + x + x^2 + x^3 + \cdots$$

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Example: find the "Mac series" for the following functions.

1-) $e^{5x} = \sum_{n=0}^{\infty} \frac{(5x)^n}{n!} = \sum_{n=0}^{\infty} \frac{5^n x^n}{n!}$

2-) $\frac{3}{e^x} = 3e^{-x} = 3\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3(-1)^n x^n}{n!}$

3-) $x e^{-3x} = x\sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{x(-3)^n x^n}{n!}$

4-) $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

5-) $\frac{1}{3-x} = \frac{1}{3(1-\frac{x}{3})} = \frac{1}{3} \cdot \frac{1}{1-\frac{x}{3}}$

$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}$

6-) $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$

$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \int x^{2k}\, dx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$

$$\boxed{\therefore \tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}}$$

تشبه sinx

7-) $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$

تكامل تكامل

$\int \frac{1}{1+x} = \int \sum (-1)^k x^k$

$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$

8-) $\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$

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9-) $\sin\left(\frac{\pi}{2}+x\right) = \sin\frac{\pi}{2}\cos x + \underbrace{\cos\frac{\pi}{2}}_{\text{zero}}\sin x$

$= \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

10-) $x\sin(2x^3) = x\sum_{k=0}^{\infty} \frac{(-1)^k (2x^3)^{2k+1}}{(2k+1)!}$

$= x\sum_{k=0}^{\infty} \frac{(-1)^k (2)^{2k+1} x^{3(2k+1)}}{(2k+1)!}$

$= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{(2k+1)} x^{6k+3}}{(2k+1)!} \cdot x$

11-) $2x^3\cos(2x) = 2x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2\, x^{2n+3}}{(2n)!}$

Example: find the sum of the following series.

1-) $\sum_{k=0}^{\infty} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{(1)^k}{k!} = e^1$

2-) $\sum_{k=2}^{\infty} \frac{2^k}{k!} = e^2 - (\overset{k=0}{1} + \overset{k=1}{2}) = e^2 - 3$

3-) $\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{(2k+1)}}{4^{2k+1}(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\pi}{4}\right)^{2k+1}}{(2k+1)!} = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

4-) $\sum_{k=1}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!} = \cos(\pi) - \overset{k=0}{1} = -1 - 1 = -2$